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Dynamic utility indifference valuation via convex risk measures

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Abstract: The (subjective) indifference value of a payoff in an incomplete financial market is that monetary amount which leaves an agent indifferent between buying or not buying the payoff when she always optimally exploits her trading opportunities. We study these values over time when they are defined with respect to a dynamic monetary concave utility functional, i.e., minus a dynamic convex risk measure. For that purpose, we prove some new results about families of conditional convex risk measures. We study the convolution of abstract conditional convex risk measures and show that it preserves the dynamic property of time-consistency. Moreover, we prove that the dynamic risk measure (or utility functional) associated to superhedging in a market with trading constraints is time-consistent. By combining these results, we deduce that the corresponding indifference valuation functional is again time-consistent. As an auxiliary tool, we establish a variant of the representation theorem for conditional convex risk measures which is in terms of equivalent probability measures. Since backward stochastic differential equations (BSDEs) induce time-consistent DMCUFs, we also show how the valuation approach works in a BSDE setting.

Key words: utility indifference valuation, dynamic valuation, dynamic monetary concave utility functionals, time-consistency, convolution, representation of risk measures, BSDE, superhedging under constraints, dynamic convex risk measures, incomplete markets

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1 Introduction

This paper deals with the valuation of contingent claims in incomplete financial markets. We present a dynamic utility indifference valuation approach which stems from the basic economic concept of certainty equivalent, modified and extended to accommodate the market environment (an idea introduced by Hodges and Neuberger [HN89]). The agents' attitudes towards risk are incorporated to establish preferences over risk which cannot be eliminated by trading.

More precisely, our investor's preferences at each time t are given by some utility functional $U_t : \mathbf{L}^\infty \to \mathbf{L}^\infty(\mathcal{F}_t)$. The investor has at time t an $\mathcal{F}_t$-measurable initial endowment x_t and can trade in a financial market, possibly under constraints. We denote by $\mathcal{C}_t$ the set of payoffs she can superhedge by trading during $(t, T]$ with zero initial endowment. At each time $t \in [0, T]$, the *utility indifference value* $p_t(X)$ of a payoff $X \in \mathbf{L}^\infty$ due at time T is defined implicitly by

$$\operatorname*{ess\,sup}_{g \in \mathcal{C}_t} U_t(x_t + g) = \operatorname*{ess\,sup}_{g \in \mathcal{C}_t} U_t(x_t - p_t(X) + g + X), \tag{1.1}$$

i.e., such that the agent is indifferent between buying X for the price $p_t(X)$ and not buying it, presuming she trades optimally in the market in both cases. U_t belongs to the class of *monetary concave utility functionals at time* t (MCUFs for short), which is defined axiomatically such that $-U_t$ is a ($\mathcal{F}_t$-conditional) convex risk measure. In particular, U_t is $\mathcal{F}_t$-*translation invariant* in the sense that

$$U_t(X + a_t) = U_t(X) + a_t \quad \text{for all } a_t \in \mathbf{L}^\infty(\mathcal{F}_t),$$

so that in (1.1) all $\mathcal{F}_t$-measurable quantities can be extracted. Hence

$$p_t(X) = U_t^{\mathrm{opt}}(X) - U_t^{\mathrm{opt}}(0), \tag{1.2}$$

where the operator

$$U_t^{\mathrm{opt}}(\,.\,) := \operatorname*{ess\,sup}_{g \in \mathcal{C}_t} U_t(\,.\, + g)$$

corresponds to the agent's market modified preferences when she takes into account her trading opportunities.

We show that similar as in [BEK05] U_t^{opt} is an MCUF and given by the *convolution* of U_t and the *market MCUF*; the latter is associated to $\mathcal{C}_t$ and constructed like in [FS02] with the help of the optional decomposition under constraints. A key issue is to ensure (strong) time-consistency for the dynamic behaviour of $p = (p_t)$. Therefore we study the convolution of two abstract conditional risk measures and prove that this operation preserves (strong) time-consistency. In the same general setting, we give sufficient conditions to guarantee that $p_t(X)$ lies inside the interval of arbitrage-free prices so that it could be considered as a price for X, and we investigate the structure of p when there are no trading constraints. We briefly discuss the connection to good deal bounds. In the special case where U is given by a backward stochastic differential equation (BSDE), we also describe the market DMCUF, U^{opt} and p in this way, and we show that the driver for U^{opt} is the pointwise convolution of the drivers of U and of the market DMCUF. This extends results of Rosazza Gianin [RG04] and Barrieu/El Karoui [BEK04]. Finally, because pricing and valuation in financial markets is done with the help of equivalent martingale measures, we also want a *representation* for MCUFs in terms of their concave conjugate functionals via *equivalent* probability measures.

Although various aspects of our approach have appeared before, the combined treatment of all ideas at the general and conditional level seems to be new. Most previous results are only given unconditionally for $t = 0$; this applies to the indifference valuation via risk measures in Xu [Xu05] or (briefly) in Barrieu/El Karoui [BEK05], to the construction of the market functional in Föllmer/Schied [FS02], or to the convolution in Barrieu/El Karoui [BEK05]. Some conditional results are available; Larsen/Pirvu/Shreve/Tütüncü [LPST05] treat indifference valuation for a special Φ_t, and Detlefsen/Scandolo [DS05] and Cheridito/Delbaen/Kupper [CDK05] provide similar representations for conditional convex risk measures; see section 3 for a more detailed comparison with these two papers. Jobert/Rogers [JR05] study several of the above issues in finite discrete time over a finite probability space. Barrieu/El Karoui [BEK04] discuss the convolution of DMCUFs which are given by BSDEs. However, they work with a class of BSDEs which is not general enough to incorporate the market functional of an incomplete market, which is constructed as in Bender/Kohlmann [BK04].

Our general results that convolution preserves time-consistency and that the market functional in an incomplete market with trading constraints is time-consistent seem to be new.

The paper is structured as follows. Notations and conventions are given in section 2. Section 3 introduces (dynamic) MCUFs. We state a representation theorem for MCUFs similar to [DS05] but in terms of equivalent probability measures; a closely related result can be found in [CDK05]. Some results about (strong) time-consistency inspired mainly by [Del03] are given. Section 4 introduces the convolution of general dynamic MCUFs and extends a result of [BEK05]. The proof is one application of the representation theorem of section 3. In section 5, we adapt the results of [FK97] about superhedging under constraints to our needs. We combine the above results in section 6 to prove that U^{opt} is the convolution of U and the market DMCUF given via the superhedging price. Then we show that the indifference valuation functional p is a dynamic MCUF, give conditions when it is strongly time-consistent and consistent with the no-arbitrage principle, and relate it to good deal bounds. Section 7 presents two examples. The first deals with dynamic MCUFs described by backward stochastic differential equations, and the second illustrates that a static MCUF cannot always be extended to a dynamic MCUF. The final section 8 concludes.

2 Notations and Conventions

Throughout this paper, we work with a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a fixed filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ where $T < \infty$ is a fixed finite time horizon. We assume that $\mathbb{F}$ satisfies the usual conditions of right-continuity and completeness. Hence we can and do choose for each martingale a right-continuous version with left limits (RCLL for short). For simplicity we let $\mathcal{F}_0$ be trivial and $\mathcal{F}_T = \mathcal{F}$. For $s < t$ an integral from s to t is defined on the half-open interval $(s, t]$. For $p \in [1, \infty]$, $\mathbf{L}^p(\Omega, \mathcal{G}, \mathbb{P})$ ($\mathbf{L}^p(\mathcal{G})$ or even $\mathbf{L}^p = \mathbf{L}^p(\mathcal{F})$ if no confusion is possible) denotes the space of all equivalence classes of real-valued, $\mathcal{G}$-measurable random variables with finite $L^p(\mathbb{P})$-norm, where $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$.

By $\mathbf{L}^0(\mathcal{F}_t, Y)$ we denote the set of all equivalence classes of $\mathcal{F}_t$-measurable mappings $\Omega \to Y$. An $\mathcal{F}_t$-partition is a family of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{F}_t$ whose union is Ω. The transpose of a vector z is denoted by z^* and $\mathbf{1}_A$ denotes the indicator function for a set $A \in \mathcal{F}$. $\mathcal{M}_1$ denotes the set of all probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, $\mathcal{M}_1(\mathbb{P})$ the set of all $\mathbb{Q} \in \mathcal{M}_1$ with $\mathbb{Q} \ll \mathbb{P}$ and $\mathcal{M}_1^e(\mathbb{P})$ the set of all $\mathbb{Q} \in \mathcal{M}_1(\mathbb{P})$ with $\mathbb{Q} \approx \mathbb{P}$. Unless mentioned otherwise, all (in-)equalities which involve random variables hold almost surely with respect to $\mathbb{P}$, (conditional) expectations and essential infima and suprema are taken with respect to $\mathbb{P}$, a density Z_T of some measure $\mathbb{Q} \in \mathcal{M}_1(\mathbb{P})$ is its density with respect to $\mathbb{P}$ on $\mathcal{F} = \mathcal{F}_T$ and its density process $Z = (Z_t)_{0\le t\le T}$ consists of its densities Z_t with respect to $\mathbb{P}$ on $\mathcal{F}_t$. We frequently identify $\mathbb{Q} \in \mathcal{M}_1(\mathbb{P})$ with its density $Z_T \in \mathbf{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. When we say that a set $\mathcal{Q} \subseteq \mathcal{M}_1(\mathbb{P})$ has a property in $\mathbf{L}^1$, we mean that the set of corresponding densities has this property. $\mathcal{Q}^e$ consists of all $\mathbb{Q} \in \mathcal{Q}$ which are equivalent to $\mathbb{P}$. We always work with equivalence classes of random variables and thus do not distinguish between different versions of, e.g., the essential infimum of a family of random variables. In particular, when defining some set depending on an equivalence class of random variables we take one (fixed) representative, in order to have that set well-defined. For the definition of processes having locally some property we refer to Definition VI.27 in [DM82]. As we consider processes on $[t, T]$ having some local properties, this definition has the advantage that a stopped process with starting point t need not have the required property on all of Ω but only on the sets $\{\tau_n > t\}$ where $(\tau_n)_{n\in\mathbb{N}}$ is a localizing sequence. In particular, for any $t \ge 0$ we have $\{\tau_n > t\} \subseteq \{\tau_n > 0\}$ and this ensures that if S is a locally bounded semimartingale on $[0, T]$, so is S on $[t, T]$. Moreover, note that the assumption of $\mathcal{F}_0$ to be trivial implies boundedness of S_0. Since we are working with a finite time horizon T, a localizing sequence for some process $(S_t)_{0\le t\le T}$ is an increasing sequence of $[0, T]$-valued stopping times τ_n, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \mathbb{P}[\tau_n < T] = 0$ and such that for each $n \in \mathbb{N}$, the stopped process S^{τ_n} has the desired property.

3 Representations and Time-Consistency of Dynamic MCUFs

In this section we introduce and study *(dynamic) monetary concave utility functionals* (MCUFs for short). This is the class of functionals we consider for utility indifference valuation in a later section. Their definition is very similar to that of convex risk measures, for which it is known that they can be equivalently described by their *acceptance set*, i.e., the set of payoffs to which they assign non-positive values. We show that there is an analogous result for (dynamic) MCUFs and also point out the differences. Then we investigate the properties, in particular continuity, of (dynamic) MCUFs. The main result of this section gives an equivalence between continuity of an MCUF, its representability, and closedness of its acceptance set. This extends well-known results from the static case to a dynamic setting. Similar dynamic results can also be found in a recent work of Detlefsen/Scandolo [DS05] and in [CDK05] in a more general setting. Finally we investigate a property called *(strong) time-consistency* which ensures that the ordering on payoffs induced by a dynamic MCUF is consistent between different points in time.

Definition 3.1 Fix $t \in [0, T]$. We call a mapping $\Phi_t : \mathbf{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbf{L}^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$ a *monetary concave utility functional at time* t (MCUF for short) if it satisfies

A) *Monotonicity:* $\Phi_t(X_1) \le \Phi_t(X_2)$ for all $X_1, X_2 \in \mathbf{L}^\infty$ with $X_1 \le X_2$.
B) $\mathcal{F}_t$*-translation invariance:* $\Phi_t(X + a_t) = \Phi_t(X) + a_t$ for all $X \in \mathbf{L}^\infty$ and $a_t \in \mathbf{L}^\infty(\mathcal{F}_t)$.
C) *Concavity:* $\Phi_t(\beta X_1 + (1-\beta)X_2) \ge \beta\Phi_t(X_1) + (1-\beta)\Phi_t(X_2)$ for all $X_1,\ X_2 \in \mathbf{L}^\infty$ and $\beta \in [0,1]$.

We say that an MCUF Φ_t is *normalized* if $\Phi_t(0) = 0$, and we call it a *monetary coherent utility functional at time* t (MCohUF for short) if it satisfies

D) *Positive homogeneity:* $\Phi_t(\lambda X) = \lambda\Phi_t(X)$ for all $X \in \mathbf{L}^\infty$ and $\lambda \ge 0$.

If Φ_t is an MCUF (respectively an MCohUF) at each time $t \in [0,T]$, we call the family $\Phi = (\Phi_t(\,.\,))_{0\le t\le T}$ a *dynamic MCUF* (respectively a *dynamic MCohUF*) and use the abbreviation DMCUF (respectively DMCohUF).

An additional property one might require of Φ_t is

E) $\mathcal{F}_t$*-regularity:* $\Phi_t(\mathbf{1}_A X_1 + \mathbf{1}_{A^c} X_2) = \mathbf{1}_A\Phi_t(X_1) + \mathbf{1}_{A^c}\Phi_t(X_2)$ for all $X_1, X_2 \in \mathbf{L}^\infty$ and $A \in \mathcal{F}_t$.

But M. Kupper has pointed out to us that monotonicity and translation invariance already imply E) as follows; see also Proposition 3.3 of [CDK05]. First of all, we have $\mathbf{1}_A\Phi_t(X\mathbf{1}_A) = \mathbf{1}_A\Phi_t(X)$ for $X \in \mathbf{L}^\infty$ and $A \in \mathcal{F}_t$, because A) and B) yield

$$\mathbf{1}_A\Phi_t(X) \underset{\ge}{\le} \mathbf{1}_A\Phi_t\big(X\mathbf{1}_A \pm \|X\|_{\mathbf{L}^\infty}\mathbf{1}_{A^c}\big) = \mathbf{1}_A\Phi_t(X\mathbf{1}_A).$$

Applying this to $X = \mathbf{1}_A X_1 + \mathbf{1}_{A^c} X_2$ gives

$$\Phi_t(X) = \mathbf{1}_A\Phi_t(X\mathbf{1}_A) + \mathbf{1}_{A^c}\Phi_t(X\mathbf{1}_{A^c}) = \mathbf{1}_A\Phi_t(X_1) + \mathbf{1}_{A^c}\Phi_t(X_2).$$

Remark 3.2 i) If Φ_t is an MCUF at time t, then $-\Phi_t$ is almost an $\mathcal{F}_t$-conditional convex risk measure in the sense of [DS05]. More precisely, these authors assume that Φ_t is normalized, impose instead of C) the stronger notion of $\mathcal{F}_t$*-concavity*, where $\beta \in \mathbf{L}^0(\mathcal{F}_t; [0,1])$, and use this to deduce that E) then automatically holds; see their Proposition 1. In addition, for MCohUFs they use instead of D) the stronger notion of $\mathcal{F}_t$*-positive homogeneity*, where $\lambda \in \mathbf{L}^0(\mathcal{F}_t; [0,\infty))$. We shall see that most of the MCUFs respectively MCohUFs we consider automatically satisfy these stronger properties.

ii) Since $\mathcal{F}_0$ is trivial, $-\Phi_0$ is simply a convex risk measure in the usual sense; see [FS04] for an comprehensive textbook account. We call $t = 0$ the *static* or *unconditional* case.

iii) In the literature, extensions from static to dynamic risk measures have been considered under two aspects. What we present here corresponds to the study of risk measures conditioned on some information. A second aspect is to define risk measures for payoff streams, i.e., on stochastic processes instead of random variables; see [Wan99], [Del03], [Det03], [Sca03], [Web03], [ADEHK04], [CDK04], [PR04], [Rie04], [CDK05] or [CDK05a] for work on that topic. Despite the importance of the latter aspect, we restrict our considerations here to the first one.

iv) We note that MCohUFs are always normalized. Moreover, under the assumption of positive homogeneity, concavity is equivalent to

F) *Superadditivity:* $\Phi_t(X_1 + X_2) \ge \Phi_t(X_1) + \Phi_t(X_2)$ for all $X_1,\ X_2 \in \mathbf{L}^\infty$. ◊

An MCUF at time $t \leq T$ assigns to each discounted net payoff X due at time T another random variable $\Phi_t(X)$. We interpret $\Phi_t(X)$ as the (individual) utility, expressed in monetary units, that some agent assigns to X at time t. However, this does not imply that it is always possible to swap at time t the future payoff X for $\Phi_t(X)$ monetary units. In fact, this would require the existence of another agent who is willing to pay $\Phi_t(X)$ in exchange for the entitlement to X. Such an agent need not exist in general.

For an economic interpretation of the axioms, we assume that there is a non-risky investment opportunity where the agent can borrow or invest arbitrary amounts of money. Moreover, we assume that all payoffs are already discounted with respect to this non-risky asset. Then the interpretation of $\mathcal{F}_t$-translation invariance is of particular interest because it clarifies the idea behind the definition of $\Phi_t(X)$ and justifies the terminology of a *monetary* utility functional; see also [FS04]. In fact, it implies that $\Phi_t(X - \Phi_t(X)) = 0$. Hence $\Phi_t(X)$ is the maximal monetary amount that can be subtracted from X at time t such that the agent still assigns a non-negative utility to the resulting (discounted) payoff $X - \Phi_t(X)$ due at time T. (To be precise, the agent cannot take the money away from X; she must borrow it from the non-risky investment and pay this debt back at time T, thus changing the discounted payoff due at time T to $X - \Phi_t(X)$.)

We emphasize that translation invariance distinguishes the considered class of utility functionals from von Neumann-Morgenstern utility functionals, most of which do not have this property. In contrast, the economic interpretation of the other axioms is more familiar. The meaning of monotonicity is obvious, and concavity models the idea that diversification should not decrease the utility. The condition that $\Phi_t(X)$ is $\mathcal{F}_t$-measurable means that values only depend on information which is available at time t. $\mathcal{F}_t$-regularity implies that an event which can already be ruled out at time t does not influence the value of $\Phi_t(X)$. As utility may grow in a non-linear way with the size of the payoff, we usually do not insist on positive homogeneity.

The issue of normalization is a bit more subtle. It depends on the exact interpretation of the random variable X to which Φ_t is applied whether this assumption makes sense or not. If X expresses a change in wealth, assuming normalization seems reasonable. But if X is some payoff to which we want to apply some utility, normalization might be inappropriate. To see this, suppose the agent has the possibility to trade in some financial market. Then she might obtain with zero initial endowment a position she personally considers to be strictly preferable to the payoff 0. In this situation she might very well assign non-zero utility to 0. Note that this again uses the idea that $\Phi_t(X)$ should be viewed as a subjective *value* rather than a (market) price. Finally, we point out that normalization can always be achieved by subtracting $\Phi_t(0)$ from the original functional. This changes the initial level of utility, but has no influence on the ordering induced by Φ_t. Nevertheless, we will see that subtracting $\Phi_t(0)$ to obtain a normalized functional can yield some difficulties.

Example 3.3 a) A classical example of an MCUF at time 0 is the *exponential certainty equivalent* with risk aversion γ, i.e.,

$$\Phi_0(X) := -\frac{1}{\gamma} \log \mathbb{E}\left[\exp\left(-\gamma X\right)\right] = \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P})} \left\{ \mathbb{E}_{\mathbb{Q}}[X] + \gamma h(\mathbb{Q}|\mathbb{P}) \right\},$$

where for $\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P})$ with density Z_T the functional $h(\mathbb{Q}|\mathbb{P}) := \mathbb{E}[Z_T \log Z_T]$ denotes the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$; see for instance Example 4.105

in [FS04], section 5 in [DS05] or Example 3.2 and Example 3.4 in [BEK04]. This MCUF is not coherent.

b) It is well-known and easy to verify that every non-empty set $\mathcal{Q} \subseteq \mathcal{M}_1^e(\mathbb{P})$ defines an MCohUF by

$$\Phi_t(X) := \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t]. \tag{3.1}$$

For $\mathcal{Q} = \{\mathbb{Q}\}$, this is just the conditional expectation under some $\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P})$. If $\mathcal{Q}$ is not a singleton, Φ_t can be interpreted to express the preferences of a conservative agent who is uncertain about the underlying model and hence takes into account several possible models. For an extension to the convex case, see Remark 3.18 below.

Note that since it is only taken over measures *equivalent* to $\mathbb{P}$, the $\mathbb{P}$-essential infimum in (3.1) is well-defined. This need not be the case if $\mathcal{Q}$ were to contain probability measures which are only absolutely continuous with respect to $\mathbb{P}$.

$\diamondsuit$

An elementary consequence of the axioms is that every MCUF is Lipschitz-continuous for the $\mathbf{L}^\infty$-norm with Lipschitz coefficient 1. In fact, translation invariance and monotonicity are already sufficient to obtain this property.

Lemma 3.4 *For any MCUF Φ_t at time t and any $X, Y \in \mathbf{L}^\infty$ we have*

$$\|\Phi_t(X) - \Phi_t(Y)\|_{\mathbf{L}^\infty} \le \|X - Y\|_{\mathbf{L}^\infty}.$$

Proof This can be shown exactly as in the static case; see Lemma 4.3 in [FS04]. $\Box$

It is well-known from the theory of risk measures that a static MCUF Φ_0 at time 0 (or, more precisely, a static risk measure) can be equivalently described by the set of all payoffs $X \in \mathbf{L}^\infty$ which are *acceptable* in the sense of having non-negative monetary utility under Φ_0. Next we show that this also holds true for $t > 0$.

Definition 3.5 For a given MCUF Φ_t, the *acceptance set* is

$$\mathcal{A}_t := \{\, X \in \mathbf{L}^\infty \mid \Phi_t(X) \ge 0 \},$$

and elements of $\mathcal{A}_t$ are called *acceptable* (with respect to Φ_t, to be precise).

Lemma 3.6 *Let Φ_t be an MCUF at time t and $\mathcal{A}_t$ its acceptance set.*

a) *$\mathcal{A}_t$ has the following properties:*
   i) *$\mathcal{A}_t$ is non-empty and convex;*
   ii) $\operatorname{ess\,sup}\{\, m_t \in \mathbf{L}^\infty(\mathcal{F}_t) \mid -m_t \in \mathcal{A}_t\} = \operatorname{ess\,sup}\,(-\mathcal{A}_t \cap \mathbf{L}^\infty(\mathcal{F}_t)) \in \mathbf{L}^\infty$;
   iii) *$-\mathcal{A}_t$ is solid, i.e., $X \in \mathcal{A}_t$, $Y \in \mathbf{L}^\infty$ and $Y \ge X$ imply that $Y \in \mathcal{A}_t$;*
   iv) *$\mathcal{A}_t$ is $\mathcal{F}_t$-regular, i.e., $X, Y \in \mathcal{A}_t$ and $A \in \mathcal{F}_t$ implies that $\mathbf{1}_A X + \mathbf{1}_{A^c} Y \in \mathcal{A}_t$.*

b) *$\mathcal{A}_t$ is radially closed, i.e., the set*

$$\{\, \lambda \in [0,1] \mid \lambda X + (1-\lambda) Y \in \mathcal{A}_t \} \tag{3.2}$$

is closed in $[0,1]$ for any $X \in \mathcal{A}_t$, $Y \in \mathbf{L}^\infty$.

c) *If Φ_t is an MCohUF, then $\mathcal{A}_t$ is a cone containing 0.*

Proof a) We only prove ii) as the other properties follow immediately from the definition of an MCUF. For later use we show a bit more. Let $X \in \mathbf{L}^\infty$. Then

$$\begin{aligned} &\text{ess sup}\,\{m_t \in \mathbf{L}^\infty(\mathcal{F}_t) \mid X - m_t \in \mathcal{A}_t\} \\ &= \text{ess sup}\,\{m_t \in \mathbf{L}^\infty(\mathcal{F}_t) \mid \Phi_t(X - m_t) \geq 0\} \\ &= \text{ess sup}\,\{m_t \in \mathbf{L}^\infty(\mathcal{F}_t) \mid \Phi_t(X) \geq m_t\} \\ &= \Phi_t(X) \in \mathbf{L}^\infty(\mathcal{F}_t). \end{aligned}$$

b) As in the static case, this follows from Lipschitz-continuity; see Proposition 4.6 in [FS04].
c) Obvious.

□

Definition 3.7 A subset $\mathcal{B}$ of $\mathbf{L}^\infty$ satisfying the properties i) – iv) in Lemma 3.6 a) is called an *acceptable set at time t.*

Lemma 3.8 *Let $\mathcal{B} \subseteq \mathbf{L}^\infty$ be an acceptable set at time t and define a mapping on $\mathbf{L}^\infty$ by*

$$\Phi_t^{\mathcal{B}}(X) := \text{ess sup}\,\{m_t \in \mathbf{L}^\infty(\mathcal{F}_t) \mid X - m_t \in \mathcal{B}\} = \text{ess sup}\,((X - \mathcal{B}) \cap \mathbf{L}^\infty(\mathcal{F}_t))\,. \quad (3.3)$$

a) *$\Phi_t^{\mathcal{B}}$ is an MCUF at time t.*
b) *If $\mathcal{B}$ is in addition closed in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$, then $\mathcal{B}$ is the acceptance set of $\Phi_t^{\mathcal{B}}$.*
c) *If $\mathcal{B}$ is the acceptance set $\mathcal{A}_t$ of an MCUF Φ_t at time t, then $\Phi_t = \Phi_t^{\mathcal{B}}$, i.e., we can recover Φ_t from its acceptance set as $\Phi_t = \Phi_t^{\mathcal{A}_t}$.*
d) *If $\mathcal{B}$ is a cone containing 0, then $\Phi_t^{\mathcal{B}}$ is an MCohUF.*

Proof a) Properties A) – C) and the fact that $\Phi_t^{\mathcal{B}}(X) \in \mathbf{L}^\infty$ can be shown as in the static case; see Proposition 4.7 in [FS04].
b) Denote by $\mathcal{A}_t$ the acceptance set of $\Phi_t^{\mathcal{B}}$. We have to show that $\mathcal{B} = \mathcal{A}_t$. Since obviously $\mathcal{B} \subseteq \mathcal{A}_t$ it suffices to show that if $X \notin \mathcal{B}$ then $\mathbb{P}[\Phi_t^{\mathcal{B}}(X) < 0] > 0$. To this end fix a constant $c \in \mathcal{B}$ such that $c > \|X\|_{\mathbf{L}^\infty}$. (Since $-\mathcal{B}$ is solid we can always take $c = \max\{\|X\|_{\mathbf{L}^\infty}, \|Z\|_{\mathbf{L}^\infty}\} + 1$, where Z is an arbitrary element of $\mathcal{B}$). Since $\mathcal{B}$ is $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$-closed, it is in particular radially closed so that there exists $\lambda > 0$ such that also $X' := \lambda c + (1 - \lambda)X \notin \mathcal{B}$. This yields the existence of a set $A \in \mathcal{F}_t$ with $\mathbb{P}[A] > 0$ such that $\Phi_t^{\mathcal{B}}(X') < 0$ on A. Indeed, suppose $\Phi_t^{\mathcal{B}}(X') \geq 0$ almost surely. Property iv) of $\mathcal{B}$ implies that

$$\left\{\, m_t \in \mathbf{L}^\infty(\mathcal{F}_t) \mid X' - m_t \in \mathcal{B}\right\}$$

is a lattice, so that there exists ([Nev75]) an increasing sequence $(m_t^n)_{n \in \mathbb{N}} \subseteq \mathbf{L}^\infty(\mathcal{F}_t)$ such that $X' - m_t^n \in \mathcal{B}$ for each n and $\Phi_t^{\mathcal{B}}(X') = \nearrow -\lim_{n\to\infty} m_t^n$. Since $(X' - m_t^n)_{n \in \mathbb{N}}$ is a uniformly bounded sequence, it converges to $X' - \Phi_t^{\mathcal{B}}(X')$ in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$, as for any $Z \in \mathbf{L}^1$ the dominated convergence theorem implies that

$$\lim_{n\to\infty} \mathbb{E}[Z(X' - m_t^n)] \;=\; \mathbb{E}[Z(X' - \Phi_t^{\mathcal{B}}(X')].$$

But since $\mathcal{B}$ is closed in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$, we get $X' - \Phi_t^{\mathcal{B}}(X') \in \mathcal{B}$, and as $-\mathcal{B}$ is solid and $\Phi_t^{\mathcal{B}}(X') \geq 0$ this implies that also $X' \in \mathcal{B}$, which is impossible. Hence $\Phi_t^{\mathcal{B}}(X') < 0$ on some set $A \in \mathcal{F}_t$ with $\mathbb{P}[A] > 0$, so that by translation invariance

$$\Phi_t^{\mathcal{B}}((1-\lambda)X) + \lambda c = \Phi_t^{\mathcal{B}}(\lambda c + (1-\lambda)X) = \Phi_t^{\mathcal{B}}(X') < 0 \quad \text{on } A. \tag{3.4}$$

Finally, Lemma 3.4 yields

$$\|\Phi_t^{\mathcal{B}}((1-\lambda)X) - \Phi_t^{\mathcal{B}}(X)\|_{\mathbf{L}^\infty} \leq \lambda \|X\|_{\mathbf{L}^\infty},$$

so that by (3.4)

$$\Phi_t^{\mathcal{B}}(X) \leq \Phi_t^{\mathcal{B}}((1-\lambda)X) + \lambda\|X\|_{\mathbf{L}^\infty} < -\lambda c + \lambda \|X\|_{\mathbf{L}^\infty} < 0 \quad \text{on } A.$$

This proves that $X \notin \mathcal{A}_t$ and hence part b).

c) This follows immediately from the proof of Lemma 3.6 a).

d) Obviously, $\Phi_t^{\mathcal{B}}(\lambda X) = \lambda \Phi_t^{\mathcal{B}}(X)$ holds for $\lambda > 0$, since $Y \in \mathcal{B}$ implies that $\lambda Y \in \mathcal{B}$ and also $\frac{1}{\lambda} Y \in \mathcal{B}$. It remains to show that $\Phi_t^{\mathcal{B}}(0) = 0$. However, $0 \in \mathcal{B}$ yields $\Phi_t^{\mathcal{B}}(0) \geq 0$. To see that we must have equality, note that the set

$$\{m_t \in \mathbf{L}^\infty(\mathcal{F}_t) \,|\, -m_t \in \mathcal{B}\}$$

is a lattice by property iv) of $\mathcal{B}$. Now suppose that $\mathbb{P}[\Phi_t^{\mathcal{B}}(0) > 0] > 0$. Then there exist $-m_t' \in \mathcal{B} \cap \mathbf{L}^\infty(\mathcal{F}_t)$ and $\epsilon > 0$ such that for $A := \{m_t' \geq \epsilon\} \in \mathcal{F}_t$ we have $\mathbb{P}[A] > 0$. But since also $-\lambda m_t' \in \mathcal{B}$ for every $\lambda > 0$, (3.3) implies that on A we have $\lambda\epsilon \leq \lambda m_t' \leq \Phi_t^{\mathcal{B}}(0)$. For $\lambda \to \infty$ this contradicts $\Phi_t^{\mathcal{B}}(0) \in \mathbf{L}^\infty$, and so $\Phi_t^{\mathcal{B}}(0) = 0$.

□

In the static case, one has a stronger result than part b) of Lemma 3.8. In fact, if $\mathcal{B}$ is an acceptable set at time 0, then Proposition 4.7 in [FS04] proves that $\mathcal{B}$ is the acceptance set of $\Phi_0^{\mathcal{B}}$ if and only if $\mathcal{B}$ is radially closed. Hence, in the static case, the acceptance set of $\Phi_0^{\mathcal{B}}$ is the smallest radially closed acceptable set at time 0 that contains $\mathcal{B}$. It is surprising that this fails to be true in the dynamic case as the following example illustrates.

Example 3.9 Let $\mathcal{F}_t = \mathcal{F}$ be generated by a disjoint partition $(A_n)_{n\in\mathbb{N}}$ of Ω such that $\mathbb{P}[A_n] > 0$ for all $n \in \mathbb{N}$. Define

$$\mathcal{B}_0 := \left\{ -\sum_{n=1}^{N} \lambda_n \mathbf{1}_{A_n} \,\middle|\, N \in \mathbb{N},\ \lambda_n \in [0,1] \right\} + \mathbf{L}_+^\infty.$$

Let us first show that $\mathcal{B} := \overline{\mathcal{B}_0}^\infty$, the $\mathbf{L}^\infty$-norm closure of $\mathcal{B}_0$, is an acceptable set at time t and also at time 0. Properties i) and iii) are obvious. Since for any $-m_t \in \mathcal{B}$ and any $\epsilon > 0$ there exists $-m_t' \in \mathcal{B}_0$ such that $\|m_t - m_t'\|_{\mathbf{L}^\infty} < \epsilon$, we get

$$\begin{aligned}
\Phi_t^{\mathcal{B}}(0) &= \operatorname{ess\,sup} \{m_t \in \mathbf{L}^\infty(\mathcal{F}_t) \,|\, -m_t \in \mathcal{B}\} \\
&= \operatorname{ess\,sup} \{m_t' \in \mathbf{L}^\infty(\mathcal{F}_t) \,|\, -m_t' \in \mathcal{B}_0\} \\
&= 1 \in \mathbf{L}^\infty(\mathcal{F}_t),
\end{aligned}$$

because $\sup_n \mathbf{1}_{A_n} \leq \Phi_t^{\mathcal{B}}(0) \leq 1$ by the definition of the essential supremum . Analogously,

$$\Phi_0^{\mathcal{B}}(0) = \sup\{c \in \mathbb{R} | -c \in \mathcal{B}\} = \sup\{c' \in \mathbb{R} | -c' \in \mathcal{B}_0\} = 0 \in \mathbb{R} :$$

Since we only allow *finite* sums $-\sum_{n=1}^{N}\lambda_n \mathbf{1}_{A_n}$ in the definition of $\mathcal{B}_0$, the property $\mathbb{P}[A_n] > 0$ for all n implies that there exists no $c > 0$ such that $-c \in \mathcal{B}_0$. Hence $\mathcal{B}$ is an acceptable set at time 0. It also is an acceptable set at time t since it is in addition $\mathcal{F}_t$-regular as the $\mathbf{L}^\infty$-norm closure of $\mathcal{B}_0$ which itself is $\mathcal{F}_t$-regular; this is readily seen by using that $\mathcal{F}_t$ is countably generated by $(A_n)_{n\in\mathbb{N}}$.

As an $\mathbf{L}^\infty$-norm closed set, $\mathcal{B}$ is in particular radially closed. By Proposition 7 in [FS04], $\mathcal{B}$ is therefore the acceptance set of $\Phi_0^{\mathcal{B}}$. If the analogous result were true in the dynamic setting, i.e., if the acceptance set of $\Phi_t^{\mathcal{B}}$ were the smallest radially closed acceptable set at time t that contains $\mathcal{B}$, then $\mathcal{B}$ would be the acceptance set of $\Phi_t^{\mathcal{B}}$. But this is not true. In fact, $\Phi_t^{\mathcal{B}}(0) = 1$ implies by translation invariance that -1 is contained in the acceptance set of $\Phi_t^{\mathcal{B}}$, but -1 is not an element of $\mathcal{B}$ since, e.g.,

$$\left\{ X \in \mathbf{L}^\infty \,\middle|\, \| -1 - X\|_{\mathbf{L}^\infty} < \frac{1}{2} \right\}$$

is an open neighborhood of -1 which has no intersection with $\mathcal{B}_0$, so that -1 cannot be in the closure of $\mathcal{B}_0$. To see that $\mathcal{B}_0$ has an empty intersection with the above open neighborhood, we use again that all sums in the definition of $\mathcal{B}_0$ are finite. $\diamondsuit$

Remark 3.10 A characterization of acceptance sets in a general setting can be found in Proposition 3.10 of [CDK05]. $\diamondsuit$

Our next goal is now to provide a representation for an MCUF Φ_t via its concave conjugate functional, which is defined as follows.

Definition 3.11 The *concave conjugate functional* of an MCUF Φ_t at time t is the mapping $\alpha_t : \mathcal{P}_t^{\approx} \mapsto L^0\big(\mathcal{F}_t; [-\infty, +\infty)\big)$,

$$\mathbb{Q} \mapsto \alpha_t(\mathbb{Q}) := \operatorname*{ess\,inf}_{X\in\mathbf{L}^\infty} \big\{ \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \Phi_t(X) \big\}, \tag{3.5}$$

where $\mathcal{P}_t^{\approx} := \{\mathbb{Q} \in \mathcal{M}_1 \,|\, \mathbb{Q} \approx \mathbb{P} \text{ on } \mathcal{F}_t\}$ is the largest set on which the essential infimum is well-defined.

Lemma 3.12 *The concave conjugate α_t of an MCUF Φ_t at time t with acceptance set $\mathcal{A}_t$ can be written as*

$$\alpha_t(\mathbb{Q}) = \operatorname*{ess\,inf}_{X\in\mathcal{A}_t} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] \quad \text{for } \mathbb{Q} \in \mathcal{P}_t^{\approx}, \tag{3.6}$$

and it has the following σ-pasting property: If $\mathbb{Q}^n$, $n \in \mathbb{N}$, are in $\mathcal{M}_1^e(\mathbb{P})$ with density processes Z^n, if $(A_n)_{n\in\mathbb{N}}$ is an $\mathcal{F}_t$-partition of Ω and if $\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P})$ is defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} := \sum_{n=1}^\infty \mathbf{1}_{A_n} \frac{Z_T^n}{Z_t^n}$, then $\alpha_t(\mathbb{Q}) = \sum_{n=1}^\infty \mathbf{1}_{A_n} \alpha_t(\mathbb{Q}^n)$.

Proof We start by proving (3.6), i.e., by showing that

$$\operatorname*{ess\,inf}_{X'\in\mathbf{L}^\infty} \big\{ \mathbb{E}_{\mathbb{Q}}[X'|\mathcal{F}_t] - \Phi_t(X') \big\} = \operatorname*{ess\,inf}_{X'\in\mathcal{A}_t} \mathbb{E}_{\mathbb{Q}}[X'|\mathcal{F}_t].$$

As $\mathcal{A}_t \subseteq \mathbf{L}^\infty$ and Φ_t is non-negative on $\mathcal{A}_t$, we clearly have

$$\begin{aligned}\operatorname*{ess\,inf}_{X'\in\mathbf{L}^\infty}\left\{\mathbb{E}_{\mathbb{Q}}[X'|\mathcal{F}_t]-\Phi_t(X')\right\} &\le \operatorname*{ess\,inf}_{X'\in\mathcal{A}_t}\left\{\mathbb{E}_{\mathbb{Q}}[X'|\mathcal{F}_t]-\Phi_t(X')\right\}\\ &\le \operatorname*{ess\,inf}_{X'\in\mathcal{A}_t}\mathbb{E}_{\mathbb{Q}}[X'|\mathcal{F}_t].\end{aligned}$$

Conversely, translation invariance ensures that for any $X \in \mathbf{L}^\infty$ we have $\hat{X} := X - \Phi_t(X) \in \mathcal{A}_t$ and hence

$$\operatorname*{ess\,inf}_{X'\in\mathcal{A}_t}\mathbb{E}_{\mathbb{Q}}[X'|\mathcal{F}_t] \le \mathbb{E}_{\mathbb{Q}}[\hat{X}|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t]-\Phi_t(X).$$

Taking the essential infimum over all $X \in \mathbf{L}^\infty$ we obtain

$$\operatorname*{ess\,inf}_{X'\in\mathcal{A}_t}\mathbb{E}_{\mathbb{Q}}[X'|\mathcal{F}_t] \le \operatorname*{ess\,inf}_{X'\in\mathbf{L}^\infty}\left\{\mathbb{E}_{\mathbb{Q}}[X'|\mathcal{F}_t]-\Phi_t(X')\right\}.$$

For the second assertion note that $\mathbb{E}\left[\sum_{n=1}^\infty \mathbf{1}_{A_n}\frac{Z_T^n}{Z_t^n}\right] = \mathbb{E}\left[\sum_{n=1}^\infty \mathbf{1}_{A_n}\frac{1}{Z_t^n}\mathbb{E}[Z_T^n|\mathcal{F}_t]\right] = 1$. Since $\mathcal{A}_t \subseteq \mathbf{L}^\infty$, (3.6) and the dominated convergence theorem imply that

$$\begin{aligned}\alpha_t(\mathbb{Q}) &= \operatorname*{ess\,inf}_{X'\in\mathcal{A}_t}\mathbb{E}\left[\sum_{n=1}^\infty \mathbf{1}_{A_n}\frac{Z_T^n}{Z_t^n}X'\middle|\mathcal{F}_t\right]\\ &= \operatorname*{ess\,inf}_{X'\in\mathcal{A}_t}\sum_{n=1}^\infty \mathbf{1}_{A_n}\mathbb{E}_{\mathbb{Q}^n}[X'|\mathcal{F}_t]\\ &= \sum_{n=1}^\infty \mathbf{1}_{A_n}\operatorname*{ess\,inf}_{X'\in\mathcal{A}_t}\mathbb{E}_{\mathbb{Q}^n}[X'|\mathcal{F}_t]\\ &= \sum_{n=1}^\infty \mathbf{1}_{A_n}\alpha_t(\mathbb{Q}^n)\end{aligned}$$

This finishes the proof. □

In Lemma 3.8 b), we proved that if $\mathcal{B}$ is an acceptable set at time t which is closed in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$, then it is the acceptance set of $\Phi_t^{\mathcal{B}}$. We shall see that $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$-closedness of the acceptance set is equivalent to a continuity property of the corresponding MCUF. As in the static case, this continuity property will be required to obtain a structural characterization of MCUFs.

Definition 3.13 An MCUF Φ_t at time t is called *continuous from above (below)* if $\lim_{n\to\infty}\Phi_t(X_n) = \Phi_t(X)$ for any sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathbf{L}^\infty$ decreasing (increasing) to some $X \in \mathbf{L}^\infty$. (Note that monotonicity of Φ_t implies the almost sure existence of the limit.)

Like in the static case, continuity from below is stronger than continuity from above:

Lemma 3.14 *If Φ_t is an MCUF at time t and continuous from below, then it is also continuous from above.*

Proof Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbf{L}^\infty$ decreasing to some $X \in \mathbf{L}^\infty$ and define $Z_n := X_n - X$, $n \in \mathbb{N}$. With $\overline{X} := X - \Phi_t(X)$, we obtain from B) and C) that

$$\begin{aligned} 0 &= \Phi_t(\overline{X}) \\ &= \Phi_t\Big(\frac{1}{2}\big(\overline{X} + Z_n\big) + \frac{1}{2}\big(\overline{X} - Z_n\big)\Big) \\ &\geq \frac{1}{2}\Phi_t\big(\overline{X} + Z_n\big) + \frac{1}{2}\Phi_t\big(\overline{X} - Z_n\big) \\ &= \frac{1}{2}\Big(\Phi_t(X + Z_n) - \Phi_t(X) + \Phi_t(X - Z_n) - \Phi_t(X)\Big) \end{aligned}$$

so that

$$\Phi_t(X + Z_n) - \Phi_t(X) \leq \Phi_t(X) - \Phi_t(X - Z_n). \tag{3.7}$$

From this together with A), continuity from below and since $X - Z_n = 2X - X_n \nearrow X$, we obtain

$$0 \leq \Phi_t(X_n) - \Phi_t(X) \leq \Phi_t(X) - \Phi_t(X - Z_n) \searrow \Phi_t(X) - \Phi_t(X) = 0.$$

Hence $\Phi_t(X_n)$ decreases to $\Phi_t(X)$ as $n \to \infty$. □

Remark 3.15 The MCUFs in Example 3.3 a) and b) are always continuous from above. The exponential certainty equivalent is also continuous from below, but for the MCohUFs in part b) this depends on the choice of $\mathcal{Q}$; see Corollary 4.35 in [FS04]. ◊

The following Theorem 3.16 is the main result of this section. It shows that for an MCUF Φ_t, the existence of a representation via the concave conjugate functional, continuity from above, and $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$-closedness of its acceptance set $\mathcal{A}_t$ are all equivalent. A detailed discussion is given below.

Theorem 3.16 *For an MCUF Φ_t at time t with acceptance set $\mathcal{A}_t$, the following are equivalent:*

I) *Φ_t is continuous from above and $\inf_{X\in\mathcal{A}_t} \mathbb{E}_{\tilde{\mathbb{Q}}}[X] > -\infty$ for some $\tilde{\mathbb{Q}} \in \mathcal{M}_1^e(\mathbb{P})$.*

II) *Φ_t can be represented as*

$$\Phi_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}_1^e(\mathbb{P})} \Big\{\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t^0(\mathbb{Q})\Big\} \tag{3.8}$$

for a mapping $\alpha_t^0 : \mathcal{M}_1^e(\mathbb{P}) \to \mathbf{L}^0(\mathcal{F}_t; [-\infty, +\infty))$ which has the σ-pasting property.

III) *Φ_t can be represented as*

$$\Phi_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}_1^e(\mathbb{P})} \Big\{\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q})\Big\} \tag{3.9}$$

where α_t is the concave conjugate of Φ_t.

IV) *$\mathcal{A}_t$ is closed in $\sigma\left(\mathbf{L}^\infty, \mathbf{L}^1\right)$ and $\inf_{X\in\mathcal{A}_t} \mathbb{E}_{\tilde{\mathbb{Q}}}[X] > -\infty$ for some $\tilde{\mathbb{Q}} \in \mathcal{M}_1^e(\mathbb{P})$.*

If Φ_t satisfies one of the above properties and is in addition positively homogeneous, hence an MCohUF, it can be represented as

$$\Phi_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}^e} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] \tag{3.10}$$

for some set $\mathcal{Q} \subseteq \mathcal{M}_1(\mathbb{P})$ and with $\mathcal{Q}^e = \mathcal{Q} \cap \mathcal{M}_1^e(\mathbb{P}) \neq \emptyset$. $\mathcal{Q}$ can be chosen convex and closed in $\mathbf{L}^1$.

Definition 3.17 If one of the equivalent properties I) – IV) is satisfied, we say that Φ_t is *well-representable.*

Remark 3.18 i) In analogy to Example 3.3 b), it is easy to see that any functional $\Phi_t : \mathbf{L}^\infty \to \mathbf{L}^\infty(\mathcal{F}_t)$ which can be represented as in (3.8) is an MCUF at time t. This does not require the σ-pasting property of α_t^0.

ii) In II) it suffices to have the σ-pasting property only for those $\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P})$ with $\alpha_t^0(\mathbb{Q}) \not\equiv -\infty$.

iii) Note that Theorem 3.16 also allows us to *define* an MCUF at time t from a suitable mapping α_t^0 by (3.8). This is particularly useful in the coherent case where Φ_t is specified via (3.10) entirely by the set $\mathcal{Q}$; see Example 3.3 b). A similar interpretation holds in the convex case, where $\alpha_t^0(\mathbb{Q})$ is a correction term which quantifies how the model $\mathbb{Q}$ is viewed. In Example 3.3 a), $\mathbb{P}$ can be seen as a reference model and the correction term is chosen proportional to the (entropic) deviation of $\mathbb{Q}$ from $\mathbb{P}$; see also section 4.3 in [FS04].

iv) One can readily check that any well-representable MCUF satisfies even $\mathcal{F}_t$-concavity and that any well-representable MCohUF satisfies $\mathcal{F}_t$-positive homogeneity.

$\diamondsuit$

Proof of Theorem 3.16

"III) $\Rightarrow$ II):" Obvious due to Lemma 3.12.

"II) $\Rightarrow$ I):" To see continuity from above, let $(X_n)_{n\in\mathbb{N}} \subseteq \mathbf{L}^\infty$ be a uniformly bounded sequence decreasing to some $X \in \mathbf{L}^\infty$. Then

$$\begin{aligned}
\searrow - \lim_{n\to\infty} \Phi_t(X_n) &= \inf_{n\in\mathbb{N}} \left\{ \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}_1^e(\mathbb{P})} \left\{ \mathbb{E}_{\mathbb{Q}}[X_n|\mathcal{F}_t] - \alpha_t^0(\mathbb{Q}) \right\} \right\} \\
&= \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}_1^e(\mathbb{P})} \left\{ \inf_{n\in\mathbb{N}} \left\{ \mathbb{E}_{\mathbb{Q}}[X_n|\mathcal{F}_t] - \alpha_t^0(\mathbb{Q}) \right\} \right\} \\
&= \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}_1^e(\mathbb{P})} \left\{ \searrow - \lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}}[X_n|\mathcal{F}_t] - \alpha_t^0 \mathbb{Q}) \right\} \\
&= \Phi_t(X),
\end{aligned}$$

where the last equality follows from the monotone convergence theorem and (3.8). It remains to prove the existence of $\tilde{\mathbb{Q}}$ as desired. To this behalf choose a sequence $(\mathbb{Q}^n)$ in $\mathcal{M}_1^e(\mathbb{P})$ and for $\epsilon > 0$ an $\mathcal{F}_t$-partition (A_n) of Ω such that

$$-\Phi_t(0) = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{M}_1^e(\mathbb{P})} \alpha_t^0(\mathbb{Q}) = \sup_{n\in\mathbb{N}} \alpha_t^0(\mathbb{Q}^n) \le \sum_{n=1}^\infty \mathbf{1}_{A_n} \alpha_t^0(\mathbb{Q}^n) + \epsilon.$$

Define $\tilde{\mathbb{Q}} \in \mathcal{M}_1^e(\mathbb{P})$ by $\frac{d\tilde{\mathbb{Q}}}{dP} := \sum_{n=1}^\infty \mathbf{1}_{A_n} \frac{Z_T^n}{Z_t^n}$ and note that the σ-pasting property of α_t^0 gives $\alpha_t^0(\tilde{\mathbb{Q}}) + \epsilon \ge -\Phi_t(0) \in \mathbf{L}^\infty$. Using (3.6) and (3.8) yields

$$\operatorname*{ess\,inf}_{X\in\mathcal{A}_t} \mathbb{E}_{\tilde{\mathbb{Q}}}[X|\mathcal{F}_t] = \operatorname*{ess\,inf}_{X\in\mathbf{L}^\infty} \left\{ \mathbb{E}_{\tilde{\mathbb{Q}}}[X|\mathcal{F}_t] - \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}_1^e(\mathbb{P})} \left\{ \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t^0(\mathbb{Q}) \right\} \right\} \ge \alpha_t^0(\tilde{\mathbb{Q}})$$

and therefore $\inf_{X\in\mathcal{A}_t} \mathbb{E}_{\tilde{\mathbb{Q}}}[X] \ge \mathbb{E}_{\tilde{\mathbb{Q}}}[\alpha_t^0(\tilde{\mathbb{Q}})] \ge -\mathbb{E}_{\tilde{\mathbb{Q}}}[\Phi_t(0)] - \epsilon > -\infty$.

"I) ⇒ III):" First we show that "$\leq$" holds in (3.9), i.e., that

$$\Phi_t(X) \leq \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}_1^e(\mathbb{P})} \{\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q})\} \tag{3.11}$$

for all $X \in \mathbf{L}^\infty$. Indeed, for any $\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P})$ and any $X \in \mathbf{L}^\infty$, (3.5) gives

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q}) &= \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \operatorname*{ess\,inf}_{X'\in\mathbf{L}^\infty} \left\{\mathbb{E}_{\mathbb{Q}}[X'|\mathcal{F}_t] - \Phi_t(X')\right\} \\
&\geq \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - (\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \Phi_t(X)) \\
&= \Phi_t(X).
\end{aligned}$$

(3.11) follows if we take the essential infimum over all $\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P})$. Inequality (3.11) implies (3.9) if we show that for any $X \in \mathbf{L}^\infty$

$$\mathbb{E}_{\tilde{\mathbb{Q}}}[\Phi_t(X)] = \mathbb{E}_{\tilde{\mathbb{Q}}}\Big[\operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}_1^e(\mathbb{P})} \{\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q})\} \Big]. \tag{3.12}$$

Similarly to, e.g., [Det03], this will be done by exploiting the well-known representation results for the static case. To derive from Φ_t an MCUF at time 0, we define the mapping $\tilde{\Phi}_0 : \mathbf{L}^\infty \to I\!R$ by $\tilde{\Phi}_0(X) := \mathbb{E}_{\tilde{\mathbb{Q}}}[\Phi_t(X)]$. This is an MCUF at time 0, and continuous from above because Φ_t is. Hence Theorem 4.31 and Remark 4.16 of [FS04] imply that it can be represented as

$$\tilde{\Phi}_0(X) = \inf_{\mathbb{Q}\in\mathcal{M}_1(\mathbb{P})} \{\mathbb{E}_{\mathbb{Q}}[X] - \tilde{\alpha}_0(\mathbb{Q})\}, \tag{3.13}$$

where

$$\tilde{\alpha}_0(\mathbb{Q}) = \inf_{Y\in\mathbf{L}^\infty} \left\{\mathbb{E}_{\mathbb{Q}}[Y] - \tilde{\Phi}_0(Y)\right\}. \tag{3.14}$$

We argue below that $\tilde{\alpha}_0(\tilde{\mathbb{Q}}) > -\infty$, and because $\tilde{\mathbb{Q}} \in \mathcal{M}_1^e(\mathbb{P})$, this implies that we have

$$\tilde{\Phi}_0(X) = \inf_{\mathbb{Q}\in\mathcal{M}_1^e(\mathbb{P})} \{\mathbb{E}_{\mathbb{Q}}[X] - \tilde{\alpha}_0(\mathbb{Q})\}. \tag{3.15}$$

Similarly to [DS05], we show next that (3.15) remains true if we take the infimum only over all $\mathbb{Q}$ in

$$\tilde{\mathcal{Q}}_t := \left\{\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P}) \,\middle|\, \mathbb{Q}[A] = \tilde{\mathbb{Q}}[A] \text{ for all } A \in \mathcal{F}_t\right\},$$

i.e., we claim that

$$\inf_{\mathbb{Q}\in\mathcal{M}_1^e(\mathbb{P})} \{\mathbb{E}_{\mathbb{Q}}[X] - \tilde{\alpha}_0(\mathbb{Q})\} = \inf_{\mathbb{Q}\in\tilde{\mathcal{Q}}_t} \{\mathbb{E}_{\mathbb{Q}}[X] - \tilde{\alpha}_0(\mathbb{Q})\}. \tag{3.16}$$

It is clear that "$\leq$" holds, and "$\geq$" will follow once we show that

$$\tilde{\alpha}_0(\mathbb{Q}) = -\infty \quad \text{for any } \mathbb{Q} \in \Big(\mathcal{M}_1^e(\mathbb{P}) \setminus \tilde{\mathcal{Q}}_t\Big). \tag{3.17}$$

But if $\mathbb{Q} \notin \tilde{\mathcal{Q}}_t$, there exists $A \in \mathcal{F}_t$ such that $\mathbb{Q}[A] \neq \tilde{\mathbb{Q}}[A]$. As $\mathcal{F}_t$-translation invariance of Φ_t implies that $\tilde{\Phi}_0(\lambda\mathbf{1}_A) = \mathbb{E}_{\tilde{\mathbb{Q}}}[\Phi_t(\lambda\mathbf{1}_A + 0)] = \mathbb{E}_{\tilde{\mathbb{Q}}}[\lambda\mathbf{1}_A] + \mathbb{E}_{\tilde{\mathbb{Q}}}[\Phi_t(0)]$, we obtain from (3.14)

$$\begin{aligned}
\tilde{\alpha}_0(\mathbb{Q}) &\leq \inf_{\lambda\in I\!R} \left\{\mathbb{E}_{\mathbb{Q}}[\lambda\mathbf{1}_A] - \tilde{\Phi}_0(\lambda\mathbf{1}_A)\right\} \\
&= \inf_{\lambda\in I\!R} \left\{\lambda\mathbb{Q}[A] - \lambda\tilde{\mathbb{Q}}[A] - \mathbb{E}_{\tilde{\mathbb{Q}}}[\Phi_t(0)]\right\} = -\infty.
\end{aligned}$$

Hence (3.16) follows. Now we show that

$$\mathbb{E}_{\tilde{\mathbb{Q}}}[\alpha_t(\mathbb{Q})] = \tilde{\alpha}_0(\mathbb{Q}) \quad \text{for all } \mathbb{Q} \in \tilde{\mathcal{Q}}_t. \tag{3.18}$$

In fact, $\mathcal{F}_t$-regularity of Φ_t implies that the set $\{\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \Phi_t(X) \mid X \in \mathbf{L}^\infty\}$ is a lattice. This guarantees ([Nev75]) the existence of some sequence $(X_n)_{n\in\mathbb{N}} \subseteq \mathbf{L}^\infty$ such that

$$\operatorname*{ess\,inf}_{X\in\mathbf{L}^\infty} \{\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \Phi_t(X)\} =\searrow - \lim_{n\to\infty} (\mathbb{E}_{\mathbb{Q}}[X_n|\mathcal{F}_t] - \Phi_t(X_n)) \tag{3.19}$$

so that by the monotone convergence theorem

$$\begin{aligned}\mathbb{E}_{\tilde{\mathbb{Q}}}\left[\operatorname*{ess\,inf}_{X\in\mathbf{L}^\infty} \{\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \Phi_t(X)\}\right] &=\searrow - \lim_{n\to\infty} \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\mathbb{E}_{\mathbb{Q}}[X_n|\mathcal{F}_t] - \Phi_t(X_n)\right] \\ &\geq \inf_{X\in\mathbf{L}^\infty} \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \Phi_t(X)\right]; \end{aligned} \tag{3.20}$$

clearly we then even have "=" in (3.20). This together with (3.5), $\mathbb{Q} \in \tilde{\mathcal{Q}}_t$ and (3.14) yields

$$\begin{aligned}\mathbb{E}_{\tilde{\mathbb{Q}}}\left[\alpha_t(\mathbb{Q})\right] &= \inf_{X\in\mathbf{L}^\infty} \left\{\mathbb{E}_{\mathbb{Q}}[X] - \mathbb{E}_{\tilde{\mathbb{Q}}}[\Phi_t(X)]\right\} \\ &= \inf_{X\in\mathbf{L}^\infty} \left\{\mathbb{E}_{\mathbb{Q}}[X] - \tilde{\Phi}_0(X)\right\} \\ &= \tilde{\alpha}_0(\mathbb{Q})\end{aligned}$$

and hence (3.18). Combining this with (3.11), (3.18), (3.15) and (3.16) we can finish the proof of (3.12) as follows:

$$\begin{aligned}\mathbb{E}_{\tilde{\mathbb{Q}}}\left[\Phi_t(X)\right] &\leq \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}_1^e(\mathbb{P})} \{\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q})\}\right] \\ &\leq \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\operatorname*{ess\,inf}_{\mathbb{Q}\in\tilde{\mathcal{Q}}_t} \{\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q})\}\right] \\ &\leq \inf_{\mathbb{Q}\in\tilde{\mathcal{Q}}_t} \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\{\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q})\}\right] \\ &= \inf_{\mathbb{Q}\in\tilde{\mathcal{Q}}_t} \left\{\mathbb{E}_{\mathbb{Q}}[X] - \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\alpha_t(\mathbb{Q})\right]\right\} \\ &= \inf_{\mathbb{Q}\in\tilde{\mathcal{Q}}_t} \{\mathbb{E}_{\mathbb{Q}}[X] - \tilde{\alpha}_0(\mathbb{Q})\} \\ &= \tilde{\Phi}_0(X) \\ &= \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\Phi_t(X)\right].\end{aligned}$$

Finally, to see that $\tilde{\alpha}_0(\tilde{\mathbb{Q}}) > -\infty$, note that $Y - \Phi_t(Y) \in \mathcal{A}_t$ for any $Y \in \mathbf{L}^\infty$. Hence (3.14) gives

$$\tilde{\alpha}_0(\tilde{\mathbb{Q}}) = \inf_{Y\in\mathbf{L}^\infty} \mathbb{E}_{\tilde{\mathbb{Q}}}[Y - \Phi_t(Y)] \geq \inf_{X\in\mathcal{A}_t} \mathbb{E}_{\tilde{\mathbb{Q}}}[X] > -\infty.$$

"I) $\Rightarrow$ IV):" Closedness of the acceptance set can be shown as in the static case, see [FS04], Theorem 4.31, c) $\Rightarrow$ e) $\Rightarrow$ f) together with Lemma 4.20.

"IV) $\Rightarrow$ I):" To see continuity from above, let $(X_n)_{n\in I\!N}$ be a uniformly bounded sequence in $\mathbf{L}^\infty$ decreasing to some $X \in \mathbf{L}^\infty$ so that

$$\searrow - \lim_{n\to\infty} \Phi_t(X_n) = Z \tag{3.21}$$

for some $Z \in \mathbf{L}^\infty(\mathcal{F}_t)$. Then $Y_n := X_n - \Phi_t(X_n)$ converges to $X - Z$ $\mathbb{P}$-a.s. and is uniformly bounded as well. By dominated convergence, $(Y_n)_{n\in I\!N}$ thus also converges to $X - Z$ in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$. But by translation invariance, $Y_n \in \mathcal{A}_t$ for all n and $\mathcal{A}_t$ is closed in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$ so that $X - Z$ is in $\mathcal{A}_t$ as well. From this together with translation invariance and since $Z \in \mathbf{L}^\infty(\mathcal{F}_t)$, we obtain that $\Phi_t(X) \geq Z$. Hence monotonicity implies (3.21) by

$$\lim_{n\to\infty} \Phi_t(X_n) = Z \leq \Phi_t(X) = \Phi_t\big(\lim_{n\to\infty} X_n\big) \leq \lim_{n\to\infty} \Phi_t(X_n).$$

To finish the proof of Theorem 3.16, it remains to show that if Φ_t is positively homogeneous, there exists a set $\mathcal{Q} \subseteq \mathcal{M}_1^e(\mathbb{P})$ such that

$$\Phi_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t].$$

By positive homogeneity, the acceptance set $\mathcal{A}_t$ is closed under multiplication with non-negative scalars and in particular $0 \in \mathcal{A}_t$. Therefore $\alpha_t(\mathbb{Q})$ from (3.6) is $\{0, -\infty\}$-valued for each $\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P})$. Next we show that there exists $\hat{\mathbb{Q}} \in \mathcal{M}_1^e(\mathbb{P})$ such that $\alpha_t\big(\hat{\mathbb{Q}}\big) = 0$. In fact, as any MCohUF is normalized, we obtain from III) that

$$0 = \Phi_t(0) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}_1^e(\mathbb{P})} \{-\alpha_t(\mathbb{Q})\} = \inf_{n\in I\!N} \{-\alpha_t(\mathbb{Q}_n)\}$$

for some sequence $(\mathbb{Q}_n)_{n\in I\!N} \subseteq \mathcal{M}_1^e(\mathbb{P})$ (see [Nev75]). Hence there exists an $\mathcal{F}_t$-partition $(A_n)_{n\in I\!N}$ and a sequence $(\mathbb{Q}^n)_{n\in I\!N}$, $\mathbb{Q}^n \in \mathcal{M}_1^e(\mathbb{P})$ with density processes Z^n, such that

$$\sum_{n=1}^{\infty} \mathbf{1}_{A_n}\, \alpha_t(\mathbb{Q}^n) = 0;$$

this uses that each $\alpha_t(\mathbb{Q}^n)$ only takes the values 0 and $-\infty$. We define the measure $\hat{\mathbb{Q}} \in \mathcal{M}_1^e(\mathbb{P})$ via its density

$$\hat{Z}_T := \sum_{n=1}^{\infty} \mathbf{1}_{A_n} \frac{Z_T^n}{Z_t^n}.$$

Lemma 3.12 then implies that

$$\alpha_t(\hat{\mathbb{Q}}) = \sum_{n=1}^{\infty} \mathbf{1}_{A_n} \alpha_t(\mathbb{Q}^n) = 0. \tag{3.22}$$

Now fix $\mathbb{Q}' \in \mathcal{M}_1^e(\mathbb{P})$ and let

$$A := \big\{\alpha_t\left(\mathbb{Q}'\right) = 0\big\} \in \mathcal{F}_t$$

(where, as usual, we consider a fixed version of $\alpha_t(\mathbb{Q}')$). If $\hat{Z}$ and Z' denote the density processes of $\hat{\mathbb{Q}}$ and $\mathbb{Q}'$, we define a new measure $\tilde{\mathbb{Q}} \in \mathcal{M}_1^e(\mathbb{P})$ via its density $\tilde{Z}_T$ as

$$\tilde{Z}_T := \mathbf{1}_A \frac{Z'_T}{Z'_t} + \mathbf{1}_{A^c} \frac{\hat{Z}_T}{\hat{Z}_t}.$$

Then Lemma 3.12 implies that

$$\alpha_t\big(\tilde{\mathbb{Q}}\big) = \mathbf{1}_A\,\alpha_t\left(\mathbb{Q}'\right) + \mathbf{1}_{A^c}\,\alpha_t\big(\hat{\mathbb{Q}}\big) = 0.$$

Because $\mathbf{1}_A\mathbb{E}_{\mathbb{Q}'}[\,.\,|\mathcal{F}_t] = \mathbf{1}_A\mathbb{E}_{\tilde{\mathbb{Q}}}[\,.\,|\mathcal{F}_t]$ and $\alpha_t(\mathbb{Q}') = -\infty$ on A^c, we obtain

$$\mathbb{E}_{\mathbb{Q}'}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q}') \geq \mathbb{E}_{\tilde{\mathbb{Q}}}[X|\mathcal{F}_t] - \alpha_t(\tilde{\mathbb{Q}})$$

by looking separately at A and A^c. In other words, when taking the essential infimum in (3.9) it is enough to restrict attention to measures like $\tilde{\mathbb{Q}}$ that have $\alpha_t(\tilde{\mathbb{Q}}) = 0$. So if we define

$$\mathcal{Q} := \{\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P}) \,|\, \alpha_t(\mathbb{Q}) \equiv 0\}\,,$$

we obtain

$$\Phi_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t]. = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}^e} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t].$$

In order to have $\mathcal{Q}$ convex and closed in $\mathbf{L}^1$, we can replace $\mathcal{Q}$ by its $\mathbf{L}^1$-closed convex hull and recall that for convex sets the norm closure and the weak closure are the same. □

The papers [DS05] and [CDK05] contain closely related representation results; the relations and differences will be discussed below after we have introduced some additional concepts. Another representation for conditional convex risk measures can be found in Rosazza Gianin [RG04] in the context of BSDEs. In the coherent case, things become simpler; see for instance [Rie04], [RSE04] or [ADEHK04]. The recent work of Weber [Web03] is less relevant for our goals, because law-invariance does not fit well with the notion of hedging.

For comparison purposes, let us first give a slight variation of Theorem 3.16; without IV′), this is simply Theorem 1 of [DS05] in our notation.

Theorem 3.19 *For an MCUF Φ_t at time t with acceptance set $\mathcal{A}_t$, the following are equivalent:*

I′) *Φ_t is continuous from above.*

II′) *Φ_t can be represented as*

$$\Phi_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}_t^=} \left\{\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t^0(\mathbb{Q})\right\} \tag{3.23}$$

for a mapping $\alpha_t^0 : \mathcal{P}_t^= \to \mathbf{L}^0(\mathcal{F}_t; [-\infty, +\infty)$ and where $\mathcal{P}_t^= := \{\mathbb{Q} \ll \mathbb{P} \,|\, \mathbb{Q} = \mathbb{P}$ on $\mathcal{F}_t\}$.

III′) *Φ_t can be represented as*

$$\Phi_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}_t^=} \left\{\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q})\right\} \tag{3.24}$$

where α_t is the concave conjugate of Φ_t.

IV′) *$\mathcal{A}_t$ is closed in $\sigma\left(\mathbf{L}^\infty, \mathbf{L}^1\right)$.*

Definition 3.20 If one of the equivalent properties I′) – IV′) is satisfied, we say that Φ_t is *representable.*

Definition 3.21 An MCUF Φ_t at time t is called *relevant* or *sensitive* if $\mathbb{P}\big[\Phi_t(-\mathbf{1}_B) < \Phi_t(0)\big] > 0$ for any $B \in \mathcal{F}$ with $P[B] > 0$.

Proposition 3.22 *Let Φ_t be an MCUF at time t.*

a) *If Φ_t is continuous from above and relevant, there exists some $\tilde{\mathbb{Q}} \in \mathcal{M}_1^e(\mathbb{P})$ such that $\inf_{X\in\mathcal{A}_t} \mathbb{E}_{\tilde{\mathbb{Q}}}[X] > -\infty$. In particular, Φ_t is well-representable.*

b) *If Φ_t is well-representable and an MCohUF at time t, then Φ_t is relevant.*

Proof b) (3.10) gives $\Phi_t(-\mathbf{1}_B) \leq -\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_B|\mathcal{F}_t]$ for some $\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P})$, and $\Phi_t(0) = 0$. Hence Φ_t is relevant.

a) Almost like in the proof of Theorem 3.16, " I) $\Longrightarrow$ III)", we define and represent an MCUF $\overline{\Phi}_0$ at time 0 by

$$\overline{\Phi}_0(X) := \mathbb{E}[\Phi_t(X)] = \inf_{\mathbb{Q}\in\mathcal{M}_1(\mathbb{P})} \left\{\mathbb{E}_{\mathbb{Q}}[X] - \overline{\alpha}_0(\mathbb{Q})\right\} = \inf_{\substack{\mathbb{Q}\in\mathcal{M}_1(\mathbb{P}),\\ \overline{\alpha}_0(\mathbb{Q})>-\infty}} \left\{\mathbb{E}_{\mathbb{Q}}[X] - \overline{\alpha}_0(\mathbb{Q})\right\} \quad (3.25)$$

with

$$\overline{\alpha}_0(\mathbb{Q}) = \inf_{Y\in\mathbf{L}^\infty} \left\{\mathbb{E}_{\mathbb{Q}}[Y] - \overline{\Phi}_0(Y)\right\};$$

the last equality in (3.25) holds since $\overline{\Phi}_0$ is finite-valued. Because Φ_t is relevant, so is $\overline{\Phi}_0$. To construct $\tilde{\mathbb{Q}} \in \mathcal{M}_1^e(\mathbb{P})$ with

$$\overline{\alpha}_0(\tilde{\mathbb{Q}}) > -\infty, \quad (3.26)$$

we define $B \in \mathcal{F}$ up to nullsets by

$$\mathbf{1}_B := \text{ess sup}\left\{\mathbf{1}_{\{Z_T^{\mathbb{Q}}>0\}} \,\middle|\, \mathbb{Q} \in \mathcal{M}_1(\mathbb{P}) \text{ and } \overline{\alpha}_0(\mathbb{Q}) > -\infty\right\}.$$

By the definition of B, for $\mathbb{Q} \in \mathcal{M}_1(\mathbb{P})$ with $\overline{\alpha}_0(\mathbb{Q}) > -\infty$, we must have $\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{B^c}] = 0$, so that by (3.25) we have $\overline{\Phi}_0(-\mathbf{1}_{B^c}) = \overline{\Phi}_0(0)$. Hence $\mathbb{P}[B] = 1$ by relevance of $\overline{\Phi}_0$. Now choose $\mathbb{Q}^n \in \mathcal{M}_1(\mathbb{P})$ with density processes Z^n and $\overline{\alpha}_0(\mathbb{Q}^n) > -\infty$ such that $\sup_{n\in I\!N} \mathbf{1}_{\{Z_T^n>0\}} = \mathbf{1}_B = 1$ $\mathbb{P}$-a.s., and $\beta_n > 0$ with $\sum_{n=1}^\infty \beta_n = 1$ and $\sum_{n=1}^\infty \beta_n\overline{\alpha}_0(\mathbb{Q}^n) > -\infty$. Then $\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} := \sum_{n=1}^\infty \beta_n Z_T^n$ defines a measure $\tilde{\mathbb{Q}} \in \mathcal{M}_1^e(\mathbb{P})$ which satisfies (3.26). With the same arguments as for (3.17) and (3.18) one can first prove that $\overline{\alpha}_0(\mathbb{Q}) = -\infty$ for any $\mathbb{Q} \in (\mathcal{M}_1^e(\mathbb{P}) \setminus \mathcal{P}_t^=)$ which implies that $\tilde{\mathbb{Q}} \in \mathcal{P}_t^=$ and then conclude that $\overline{\alpha}_0(\tilde{\mathbb{Q}}) = \mathbb{E}_{\mathbb{P}}[\alpha_t(\tilde{\mathbb{Q}})]$ so that (3.6) yields

$$\inf_{X\in\mathcal{A}_t} \mathbb{E}_{\tilde{\mathbb{Q}}}[X] \geq \mathbb{E}_{\tilde{\mathbb{Q}}}\Big[\operatorname*{ess\,inf}_{X\in\mathcal{A}_t} \mathbb{E}_{\tilde{\mathbb{Q}}}[X|\mathcal{F}_t]\Big] = \mathbb{E}_{\mathbb{P}}[\alpha_t(\tilde{\mathbb{Q}})] = \overline{\alpha}_0(\tilde{\mathbb{Q}}) > -\infty.$$

Hence $\tilde{\mathbb{Q}}$ does the job. □

Now we can discuss the differences between the representation results in [DS05], [CDK05] and Theorem 3.16. Both [DS05] and [CDK05] work with normalized conditional risk measures. However, neither the change of sign nor the normalization are very important; for the latter note that for the normalized MCUF $\Phi_t' = \Phi_t - \Phi_t(0)$ we have $\mathcal{A}_t' = \mathcal{A}_t + \Phi_t(0)$ and $\alpha_t'(\mathbb{Q}) = \alpha_t(\mathbb{Q}) + \Phi_t(0)$.

Let us first look at [DS05]. Condition IV) does not explicitly appear there, but it follows as in the above proof of Theorem 3.16. The main difference between the two representations in Theorem 3.16 and Theorem 3.19 (which is due to [DS05]) lies in the set of representing measures. In contrast to [DS05] who use $\mathcal{P}_t^=$, we work with $\mathcal{M}_1^e(\mathbb{P})$ which does not depend on t and, more importantly, only contains measures which are *equivalent* to $\mathbb{P}$. The price for this is our additional assumption that $\inf_{X \in \mathcal{A}_t} \mathbb{E}_{\tilde{\mathbb{Q}}}[X] > -\infty$ for some $\tilde{\mathbb{Q}} \in \mathcal{M}_1^e(\mathbb{P})$.

Cheridito/Delbaen/Kupper study in [CDK05] dynamic risk measures which are defined on discrete-time processes and not like here only on random variables X. Hence there notation is different from the one use in [DS05] and here. One could specialize their Corollary 3.23 to obtain a representation for MCUFs on random variables, even in terms of equivalent measures. However, their Corollary 3.23 contains a relevance condition similar to the one in Definition 3.21; see also Definition 4.32 and Corollaries 4.31 and 9.30 in [FS04] for this economically very natural concept. As shown in Proposition 3.22, relevance is sufficient for our assumption (in I) of Theorem 3.16) that $\inf_{X \in \mathcal{A}_t} \mathbb{E}_{\tilde{\mathbb{Q}}}[X] > -\infty$ for some $\tilde{\mathbb{Q}} \in \mathcal{M}_1^e(\mathbb{P})$. But unless Φ_t is coherent, relevance is (unlike our latter assumption) in general not necessary for Theorem 3.16. In this sense our result is a little more precise.

None of the properties imposed on DMCUFs so far requires any relation between the MCUFs at different points in time. To actually study the *dynamic* behavior of DMCUFs, we introduce a notion of time-consistency.

Definition 3.23 A DMCUF $\Phi := (\Phi_t)_{0 \le t \le T}$ is called *time-consistent* if for $X, Y \in \mathbf{L}^\infty$ and $s \le t$,

$$\Phi_t(X) = \Phi_t(Y) \qquad \text{implies that} \qquad \Phi_s(X) = \Phi_s(Y). \tag{3.27}$$

Φ is called *strongly time-consistent* if in addition $\mathcal{A}_t \subseteq \mathcal{A}_s$ for $t \ge s$.

In the literature, one can find several differing definitions of time-consistency; see for instance [Pen04], [Web03], or [ADEHK04] for an overview. For our purposes, (3.27) means that indifference at time t between two payoffs X and Y is carried "forward" to any time $s < t$, i.e., when less information is available. Because the "=" sign could obviously be replaced by "$\geq$" signs in (3.27), time-consistency preserves the ordering between payoffs over time, but does not fix the level at which this occurs. Unless all Φ_t are normalized, (3.27) therefore does not guarantee that an X acceptable in t is also acceptable at time $s < t$; this requires strong time-consistency. We do not impose normalization here since we later consider operations on DMCUFs which preserve (strong) time-consistency, but may change the initial utility level; see the remark after Theorem 4.3.

Remark 3.24 i) In section 7.1 we investigate DMCUFs which are defined via solutions of backward stochastic differential equations. As they are always time-consistent, these provide us with a big class of examples for time-consistent DMCUFs.

ii) Epstein and Schneider's Example 4.1 in [ES03] illustrates that under ambiguity aversion, a rational agent might well exhibit a time-inconsistent behavior. Like for all axioms concerning decision making, it is thus important to be aware of situations where seemingly natural rules are violated.

$\Diamond$

For a DMCUF $(\Phi_t)_{0 \le t \le T}$ with acceptance sets $(\mathcal{A}_t)_{0 \le t \le T}$ and for $s \le t$, we use the notation $\mathcal{A}_s(\mathcal{F}_t) := \mathcal{A}_s \cap \mathbf{L}^\infty(\mathcal{F}_t)$. We note that $\Phi_{s \circ t} := \Phi_s \circ \Phi_t$ is an MCUF at time s and denote by $\mathcal{A}_{s \circ t}$ its acceptance set. Similarly as in Theorems 6.2 and 7.9 in [Del03], time-consistency can then be characterized as follows; see also Proposition 8 of [DS05].

Lemma 3.25 *For a DMCUF* $\Phi = (\Phi_t)_{0 \le t \le T}$*, the properties*

a) $\Phi_s = \Phi_{s \circ t}$ *for all* $s \le t$*,*

b) $\mathcal{A}_s = \mathcal{A}_{s \circ t}$ *for all* $s \le t$*,*

c) $\mathcal{A}_s = \mathcal{A}_s(\mathcal{F}_t) + \mathcal{A}_t$ *for all* $s \le t$*,*

are all equivalent and imply

d) Φ *is time-consistent.*

If Φ *is normalized, i.e.,* $\Phi_t(0) \equiv 0$ *for all* $t \in [0, T]$*, then d) is equivalent to a) – c).*

Proof a) implies d) and by c) of Lemma 3.8 is equivalent to b). If $\Phi_t(0) \equiv 0$, take $X \in \mathbf{L}^\infty$ and define $Y := \Phi_t(X)$ to get by translation invariance $\Phi_t(Y) = \Phi_t\big(0 + \Phi_t(X)\big) = \Phi_t(X)$. Time-consistency then yields $\Phi_s(X) = \Phi_s(Y) = \Phi_{s \circ t}(X)$ so that d) implies a).

"b) $\Rightarrow$ c)": To show the inclusion "$\supseteq$", let $X = X_1 + X_2$ with $X_1 \in \mathcal{A}_s(\mathcal{F}_t)$, $X_2 \in \mathcal{A}_t$ and use translation invariance and $X_2 \in \mathcal{A}_t$ to get $\Phi_t(X) = X_1 + \Phi_t(X_2) \ge X_1$. Monotonicity and $X_1 \in \mathcal{A}_s(\mathcal{F}_t)$ thus yield $\Phi_{s \circ t}(X) \ge \Phi_s(X_1) \ge 0$ so that $X \in \mathcal{A}_{s \circ t} = \mathcal{A}_s$ by b). For the converse inclusion, write $X \in \mathcal{A}_s$ as $X = \Phi_t(X) + \big(X - \Phi_t(X)\big)$. The second summand is in $\mathcal{A}_t$, and the first is in $\mathcal{A}_s(\mathcal{F}_t)$ since $\Phi_s\big(\Phi_t(X)\big) = \Phi_{s \circ t}(X) \ge 0$ because $X \in \mathcal{A}_s = \mathcal{A}_{s \circ t}$ by b).

"c) $\Rightarrow$ b)": To show "$\subseteq$", write $X \in \mathcal{A}_s$ by c) as $X = X_1 + X_2$ with $X_1 \in \mathcal{A}_s(\mathcal{F}_t)$ and $X_2 \in \mathcal{A}_t$. As above, this yields $\Phi_t(X) \ge X_1$ and hence by monotonicity of Φ_s that $\Phi_s\big(\Phi_t(X)\big) \ge \Phi_s(X_1) \ge 0$ since $X_1 \in \mathcal{A}_s$. Thus $\Phi_t(X) \in \mathcal{A}_s$ which is equivalent to $X \in \mathcal{A}_{s \circ t}$. To obtain "$\supseteq$", note that $X \in \mathcal{A}_{s \circ t}$ gives $\Phi_t(X) \in \mathcal{A}_s(\mathcal{F}_t)$ so that $X = \big(X - \Phi_t(X)\big) + \Phi_t(X) \in \mathcal{A}_t + \mathcal{A}_s(\mathcal{F}_t) = \mathcal{A}_s$ by c). $\square$

For a normalized DMCUF, time-consistency and strong time-consistency are the same. In fact, $\Phi_s(0) = 0$ implies $0 \in \mathcal{A}_s(\mathcal{F}_t)$ and therefore $\mathcal{A}_t \subseteq \mathcal{A}_s$ by c) of Lemma 3.25. Moreover, each of the equivalent properties a) – c) in Lemma 3.25 implies that Φ is normalized. To see this for a), simply write $\Phi_{t \circ t}(0) = \Phi_t\big(0 + \Phi_t(0)\big) = \Phi_t(0) + \Phi_t(0) = 2\Phi_t(0)$. Moreover, an arbitrary DMCUF $\Phi := (\Phi_t)_{0 \le t \le T}$ is time-consistent if and only if the normalized DMCUF $\Phi' := (\Phi'_t)_{0 \le t \le T}$ defined by $\Phi'_t(\,.\,) := \Phi_t(\,.\,) - \Phi_t(0)$ is (strongly) time-consistent. The acceptance set of Φ'_t is $\mathcal{A}'_t := \mathcal{A}_t + \Phi_t(0)$, where $\mathcal{A}_t$ denotes the acceptance set of Φ_t.

Suppose that a DMCUF Φ satisfies $\mathcal{A}_t \subseteq \mathcal{A}_s$ for $t \ge s$. Then $t \mapsto \inf_{X \in \mathcal{A}_t} \mathbb{E}_{\tilde{\mathbb{Q}}}[X]$ is increasing and thus $\inf_{X \in \mathcal{A}_t} \mathbb{E}_{\tilde{\mathbb{Q}}}[X] > -\infty$ holds for all t as soon as we have this for $t = 0$, i.e., if $\alpha_0(\tilde{\mathbb{Q}}) = \inf_{X \in \mathcal{A}_0} \mathbb{E}_{\tilde{\mathbb{Q}}}[X] > -\infty$. Hence condition I) in Theorem 3.16 simplifies in this case. Similarly, a time-consistent DMCUF Φ with Φ_0 relevant has Φ_t relevant for all t.

For the economic interpretation of property a) in Lemma 3.25, note that $\Phi_t(X) = \Phi_t(\Phi_t(X))$ for any normalized DMCUF Φ. This means that the agent assigns at time t the same monetary utility to X and to $\Phi_t(X)$. If she acts in a time-consistent way, she should stick to this indifference at time s, which yields exactly property a). Clearly property b) is just a reformulation of a). For property c), we note that $\mathcal{A}_s \subseteq \mathcal{A}_s(\mathcal{F}_t) + \mathcal{A}_t$ means that we can split any payoff acceptable at time s into the sum of a payoff X_1 which is acceptable at time s when the observation period ends at time t, and a payoff X_2 which is acceptable if the observation period starts at time t. Conversely, let a payoff X be the sum of such X_1 and X_2. Normalization and translation invariance imply that $\Phi_t(X_1+X_2) = \Phi_t(X_2) + X_1 \geq X_1 = \Phi_t(X_1)$, i.e., at time t the agent prefers the payoff $X = X_1 + X_2$ to X_1. If she acts in a time-consistent way, she should have the same ordering at time s; see the comment after Definition 3.23. As X_1 is acceptable at time s, this shows that the converse inclusion should hold as well. Note that the above interpretations all use that Φ is normalized.

Remark 3.26 Until now, we have considered DMCUFs for the time horizon T. To emphasize the dependence on T we write

$$\Phi_{s,T}(\,.\,) \text{ instead of } \Phi_s(\,.\,).$$

In view of a possible study of utility indifference valuation functionals for intermediate time horizons $t < T$ one could also look at DMCUFs $(\Phi_{s,t}(\,.\,))_{0\leq s\leq t}$ for all $t < T$, the idea being that $\Phi_{s,t}(X)$ is the value at time s for the payoff $X \in \mathbf{L}^\infty(\mathcal{F}_t)$ due at time t (instead of T). Where such a $\Phi_{\,.\,,t}$ comes from will be discussed later. In general, a property one might want to have for such families of functionals (in addition to (strong) time-consistency of $\Phi_{\,.\,,T}$) is

($\mathcal{R}$) *Recursiveness:* $\Phi_{s,t}(\Phi_{t,T}(X)) = \Phi_{s,T}(X)$ for any $X \in \mathbf{L}^\infty(\mathcal{F}_T)$.

(This could also be called Bellman's principle.) Note the difference between recursiveness and property a) in Lemma 3.25, where we have $\Phi_{s,T}$ instead of $\Phi_{s,t}$. In economic terms, recursiveness means that if we want to value the time T payoff X at time s, we can either do this directly or first value it at time $t \geq s$ and then value that result at time s. This can also be desirable for non-normalized functionals. The concept of recursiveness seems to go back to Peng who studied it in the context of non-linear expectations; see [Pen04] for a comprehensive overview. The following considerations are motivated by a discussion with S. Peng.

The aim of the present work is to obtain a valuation functional from utility indifference considerations. Among other things, we assume that there exists a bank account with zero interest rate, so that money can be freely transferred over time. Hence an investor should be indifferent between receiving a payoff $X \in \mathbf{L}^\infty(\mathcal{F}_t)$ at time $t < T$ or at time T. If the utility indifference valuation functionals over time are given by a family p, we should therefore have

$$p_{s,t}(X) = p_{s,T}(X) \quad \text{for all } s \leq t \leq T \text{ and } X \in \mathbf{L}^\infty(\mathcal{F}_t). \tag{3.28}$$

In addition, indifference valuation functionals should be normalized, i.e., $p_{s,t}(0) = 0$ for all $s \leq t \leq T$. With this and (3.28), time-consistency and recursiveness are equivalent; see Lemma 3.25. Moreover, ($\mathcal{R}$) for p then also holds for any time horizon $u \geq t$ instead of T.

The utility indifference valuation DMCUF $p_{\,.\,,T}$ will be obtained from a (strongly) time-consistent DMCUF $\Phi_{\,.\,,T}$ via normalization, i.e., $p_{s,T}(\,.\,) = \Phi_{s,T}(\,.\,) - \Phi_{s,T}(0)$. Here difficulties can arise if we want to valuate also for intermediate time horizons $t < T$ but do not start with normalized families Φ. In fact, for all $s \leq t \leq T$ we want to have MCUFs (with time horizon t) $\Phi_{s,t} : \mathbf{L}^\infty(\mathcal{F}_t) \to \mathbf{L}^\infty(\mathcal{F}_s)$ and then to set

$$p_{s,t}(X) := \Phi_{s,t}(X) - \Phi_{s,t}(0) \quad \text{for all } X \in \mathbf{L}^\infty(\mathcal{F}_t). \tag{3.29}$$

With this construction, we can assume (3.28) if and only if

$$\Phi_{s,t}(X) - \Phi_{s,t}(0) = \Phi_{s,T}(X) - \Phi_{s,T}(0) \quad \text{for all } s \leq t \leq T \text{ and } X \in \mathbf{L}^\infty(\mathcal{F}_t). \tag{3.30}$$

This holds, e.g., if the utility indifference valuation DMCUF is constructed from the conditional exponential certainty equivalent; see Example 7.19. If Φ satisfies (3.30) and $\Phi_{\cdot,T}$ is time-consistent, then $\Phi_{\cdot,t}$ is also time-consistent for each t. $p_{\cdot,t}$ from (3.29) is then normalized and strongly time-consistent for each t, and hence the family p also satisfies $(\mathcal{R})$.

Since the family Φ is the basic building block in the above construction, we now have to ask where $\Phi_{\cdot,t}$ comes from. $\Phi_{\cdot,T}$ is always given, and the simplest way to obtain some $\Phi_{\cdot,t}$ satisfying (3.30) is the brute force definition

$$\Phi_{s,t}(X) := \Phi_{s,T}(X) \quad \text{for } s \le t \le T \text{ and } X \in \mathbf{L}^\infty(\mathcal{F}_t).$$

This will always work, but is not always reasonable. Suppose for instance that $\Phi_{s,t}$ should represent some maximal subjective utility achievable between s and t. Then another reasonable definition could be

$$\Phi_{s,t}(X) := \Phi_{s,T}(X) - \Phi_{t,T}(0) \quad \text{for all } s \le t < T \text{ and } X \in \mathbf{L}^\infty(\mathcal{F}_t). \tag{3.31}$$

The loose argument for subtracting the second term is that since X is known at time t, it is by translation invariance irrelevant for the maximal utility achievable during the period from t to T. (But of course such an "argument" via splitting $(s,T]$ into $(s,t]$ and $(t,T]$ is based on the intuition from recursiveness and thus has a taste of circularity.) It is straightforward to check that (3.31) implies (3.30). However, $\Phi_{s,t}(X)$ is not $\mathcal{F}_s$-measurable unless $\Phi_{t,T}(0)$ is, and if this should hold for all s, we must require that $(\Phi_{s,T}(0))_{0\le s\le T}$ is a deterministic process. In that case, (3.31) gives a good definition.

In section 7.1, we shall examine functionals $\Phi_{\cdot,T}$ defined via backward stochastic differential equations (BSDEs). In that case, the BSDE also produces a natural definition for $\Phi_{\cdot,t}$ for each $t < T$, and one can show that if $\Phi_{\cdot,T}(0)$ is deterministic, those $\Phi_{\cdot,t}$ must be of the form (3.31). In that sense, this definition is also natural. $\diamondsuit$

Although time-consistency is desirable in most situations, it is also quite restrictive as we shall illustrate by an example in section 7.2. In preparation and to complete the results here, we provide another equivalent description of time-consistency for the case where the DMCUF is coherent. Since DMCohUFs are always normalized, this description is also equivalent to strong time-consistency.

Definition 3.27 A set $\mathcal{Q} \subseteq \mathcal{M}_1(\mathbb{P})$ is called *weakly multiplicatively stable* (*weakly m-stable* for short) if it contains $\mathbb{P}$ and has the following property: If we take any $\mathbb{Q}^1, \mathbb{Q}^2 \in \mathcal{Q}$ with associated density processes Z^1, Z^2, fix $t \in [0,T]$, impose that $\mathbb{Q}^2 \in \mathcal{M}_1^e(\mathbb{P})$ and define

$$Z_T := \frac{Z_t^1}{Z_t^2} Z_T^2,$$

then Z_T is the density of some element in $\mathcal{Q}$.

Remark 3.28 i) Intuitively, weak m-stability means that $\mathcal{Q}$ is closed under the following operation: We pick any time t and construct from $\mathbb{Q}^1$, $\mathbb{Q}^2 \in \mathcal{Q}$ a new probability measure $\mathbb{Q} \in \mathcal{Q}$ which agrees with $\mathbb{Q}^1$ on $\mathcal{F}_t$ and has after t the same $\mathcal{F}_t$-conditional behavior as $\mathbb{Q}^2$.

ii) Definition 3.27 is similar to the definition of m-stable sets given in [Del03]. However, we only paste together probability measures at deterministic times, whereas Delbaen also considers stopping times. Moreover, since we assumed $\mathcal{F}_0$ to be trivial, our definition is slightly simpler than the one in [Del03].

$\Diamond$

The following Lemma 3.29 is a slight improvement of Theorem 6.2 of [Del03] as it does not only give (in part a)) a structural description of time-consistent DMCohUFs of a particular form, but also shows (in part b)) that every ($\mathbb{P}$-dominated) normalized time-consistent DMCUF which is well-representable and positively homogeneous at time 0 is of this form, and gives an explicit representation. This will prove helpful in the above mentioned example of section 7.2.

Lemma 3.29 a) *Define a family of mappings $\Phi = (\Phi_t)_{0\leq t\leq T}$ on $\mathbf{L}^\infty$ by*

$$\Phi_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}^e} \mathbb{E}_{\mathbb{Q}}[X \,|\, \mathcal{F}_t] \tag{3.32}$$

for some L^1-closed and convex set $\mathcal{Q} \subseteq \mathcal{M}_1(\mathbb{P})$ with $\mathbb{P} \in \mathcal{Q}$. Then Φ is a well-representable strongly time-consistent DMCohUF if and only if $\mathcal{Q}$ is weakly m-stable. Moreover, we clearly have $\Phi_0(\,.\,) \leq \mathbb{E}_{\mathbb{P}}[\,.\,]$ on $\mathbf{L}^\infty$.

b) *Conversely, let $\Phi = (\Phi_t)_{0\leq t\leq T}$ be a normalized time-consistent DMCUF such that Φ_0 is positively homogeneous, representable and satisfies $\Phi_0(\,.\,) \leq \mathbb{E}_{\mathbb{P}}[\,.\,]$ on $\mathbf{L}^\infty$. Then Φ can be represented as in (3.32) and is in particular a DMCohUF, i.e., positively homogeneous for all $t \in [0,T]$. Moreover, $\mathcal{Q}$ is unique, weakly m-stable and consists of all $\mathbb{Q} \in \mathcal{M}_1(\mathbb{P})$ whose densities are elements of the polar cone of the acceptance set at time 0, i.e.,*

$$\mathcal{Q} = \left\{ \mathbb{Q} \in \mathcal{M}_1(\mathbb{P}) \,|\, d\mathbb{Q} = Z_T\, d\mathbb{P},\, Z_T \in \mathcal{A}_0^\circ \cap \mathcal{B}(\mathbf{L}^1) \right\}. \tag{3.33}$$

Here $\mathcal{B}(\mathbf{L}^1)$ is the unit ball in $\mathbf{L}^1$, and the polar cone of the acceptance set $\mathcal{A}_0$ at time 0 is given by

$$\mathcal{A}_0^\circ = \left\{ Z \in \mathbf{L}^1 \,\middle|\, \mathbb{E}[ZX] \geq 0 \text{ for all } X \in \mathcal{A}_0 \right\}.$$

Remark 3.30 i) The assumption $\Phi_0(\,.\,) \leq \mathbb{E}_{\mathbb{P}}[\,.\,]$ in b) is not very restrictive. It is made to ensure that $\mathbb{P}$ is in $\mathcal{Q}$, so that $\mathcal{Q}$ contains at least one element of $\mathcal{M}_1^e(\mathbb{P})$, i.e., $\mathcal{Q}^e \neq \emptyset$. However, $\mathbb{P}$ is used only to specify the null sets, and so we might equivalently demand that $\Phi_0(\,.\,) \leq \mathbb{E}_{\mathbb{Q}}[\,.\,]$ for some $\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P})$. This in turn is satisfied by any well-representable MCohUF at time 0.

ii) Lemma 3.29 shows that a DMCUF which is positively homogeneous at time 0 can only be time-consistent if the set of representing measures at time 0 or the acceptance set at time 0 (more precisely, its polar cone) has an appropriate structure. Moreover, it shows that there exists at most one normalized time-consistent DM-CUF which extends a given static MCohUF at time 0. In section 7.2 we consider an example of a static MCohUF at time 0 which can not be extended to a time-consistent DMCUF.

$\Diamond$

Proof of Lemma 3.29

a) This follows immediately from the proof of Theorem 6.2 in [Del03] if we replace all stopping times there by deterministic times. The assumption that $\mathbb{P}$ is contained in $\mathcal{Q}$ is obviously necessary from the definition of a (weakly) m-stable set.

b) By the proof of Theorem 3.2 in [Del02], with $\mathcal{Q}$ from (3.33),

$$\Phi_0(\,.\,) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\,.\,] \tag{3.34}$$

on $\mathbf{L}^\infty$, and $\mathcal{Q}$ is $\mathbf{L}^1$-closed and convex. To show that Φ_t can be represented by (3.32), we define a DMCUF $\hat{\Phi} = (\hat{\Phi}_t)_{0\le t\le T}$ as the RHS of (3.32), i.e.,

$$\hat{\Phi}_t(X) := \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}^e} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] \quad \text{for all } X \in \mathbf{L}^\infty,$$

and we show that $\Phi = \hat{\Phi}$. Then $\Phi_0(\,.\,) \le \mathbb{E}_{\mathbb{P}}[\,.\,]$ implies that $1 \in \mathcal{A}_0^\circ$ so that $\mathbb{P} \in \mathcal{Q}$, and a) implies in addition weak m-stability of $\mathcal{Q}$ since Φ is time-consistent.

Since we can replace $\mathcal{Q}$ by $\mathcal{Q}^e$ in (3.34), we have

$$\Phi_0 = \hat{\Phi}_0 \tag{3.35}$$

on $\mathbf{L}^\infty$. In fact, fix $\mathbb{Q}' \in \mathcal{Q}$ with density Z'_T and define for each $\epsilon > 0$ a measure $\mathbb{Q}^\epsilon \in \mathcal{Q}^e$ by its density $Z^\epsilon_T := \epsilon + (1-\epsilon)Z'_T$. Clearly, as ϵ tends to zero, Z^ϵ_T converges to Z'_T in $\mathbf{L}^1$ and hence also weakly in $\mathbf{L}^1$. This shows (3.35).

It is clear from Lemma 3.8 that two DMCohUFs $\big(\Phi^1_t(\,.\,)\big)_{0\le t\le T}$ and $\big(\Phi^2_t(\,.\,)\big)_{0\le t\le T}$ are equal if and only if they have the same acceptance set at each time t. Therefore we are left to show that for all $t \in (0,T]$

$$\mathcal{A}_t := \{\, X \in \mathbf{L}^\infty |\, \Phi_t(X) \ge 0 \,\} = \Big\{\, X \in \mathbf{L}^\infty(\mathcal{F}) |\, \hat{\Phi}_t(X) \ge 0 \,\Big\} =: \hat{\mathcal{A}}_t.$$

Fix $t \in (0,T]$ and let $X \in \hat{\mathcal{A}}_t$, i.e., $\hat{\Phi}_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}^e} \mathbb{E}_{\mathbb{Q}}[X\,|\,\mathcal{F}_t] \ge 0$. Then (3.35) yields

$$0 \le \inf_{\mathbb{Q}\in\mathcal{Q}^e} \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_A X] = \Phi_0(\mathbf{1}_A X) \quad \text{for all } A \in \mathcal{F}_t. \tag{3.36}$$

As Φ is time-consistent and normalized, Lemma 3.25 and $\mathcal{F}_t$-regularity imply that

$$\Phi_0\left(\mathbf{1}_A X\right) = \Phi_0(\Phi_t(\mathbf{1}_A X)) = \Phi_0\big(\mathbf{1}_A \Phi_t(X)\big) \quad \text{for all } A \in \mathcal{F}_t.$$

From this, (3.36), (3.35) and since $\mathbb{P} \in \mathcal{Q}^e$, we obtain that

$$0 \le \Phi_0\left(\mathbf{1}_A \Phi_t(X)\right) = \inf_{\mathbb{Q}\in\mathcal{Q}^e} \mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}_A \Phi_t(X)\right] \le \mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_A \Phi_t(X)\right] \quad \text{for all } A \in \mathcal{F}_t.$$

But since $\Phi_t(X)$ is $\mathcal{F}_t$-measurable, this implies that $\Phi_t(X) \ge 0$. Hence $X \in \mathcal{A}_t$ and $\hat{\mathcal{A}}_t \subseteq \mathcal{A}_t$. To show the converse inclusion, suppose that $\Phi_t(X) \ge 0$. Then $\mathcal{F}_t$-regularity and normalization yield

$$\Phi_t(\mathbf{1}_A X) = \mathbf{1}_A \Phi_t(X) \ge 0 \quad \text{for all } A \in \mathcal{F}_t.$$

Hence time-consistency, monotonicity and normalization imply that

$$\Phi_0(\mathbf{1}_A X) = \Phi_0\big(\Phi_t(\mathbf{1}_A X)\big) \ge 0 \quad \text{for all } A \in \mathcal{F}_t.$$

Consequently, we have by (3.35) that

$$\inf_{\mathbb{Q}\in\mathcal{Q}^e} \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_A X] = \Phi_0(\mathbf{1}_A X) \ge 0 \quad \text{for all } A \in \mathcal{F}_t$$

and obtain

$$\hat{\Phi}_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}^e} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] \ge 0.$$

This shows that $\mathcal{A}_t = \hat{\mathcal{A}}_t$.

We are left to show uniqueness of $\mathcal{Q}$, and it suffices to prove that there is a unique representing set at time 0. Suppose there exists another $\mathbf{L}^1$-closed and convex set $\tilde{\mathcal{Q}} \neq \mathcal{Q} \subseteq \mathcal{M}_1(\mathbb{P})$ such that

$$\Phi_0(\,.\,) = \inf_{\mathbb{Q}\in\mathcal{Q}^e} \mathbb{E}_{\mathbb{Q}}[\,.\,] = \inf_{\mathbb{Q}\in\tilde{\mathcal{Q}}^e} \mathbb{E}_{\mathbb{Q}}[\,.\,] \tag{3.37}$$

on $\mathbf{L}^\infty$. Then we apply the same arguments as in the proof of (3.35) to conclude that also

$$\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\,.\,] = \inf_{\mathbb{Q}\in\tilde{\mathcal{Q}}} \mathbb{E}_{\mathbb{Q}}[\,.\,] \tag{3.38}$$

on $\mathbf{L}^\infty$. Without loss of generality there exists $\tilde{\mathbb{Q}} \in \tilde{\mathcal{Q}}\backslash\mathcal{Q}$, and so the Hahn-Banach theorem yields some $X \in \mathbf{L}^\infty$ such that

$$\Phi_0(X) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X] > \mathbb{E}_{\tilde{\mathbb{Q}}}[X] \geq \inf_{\mathbb{Q}\in\tilde{\mathcal{Q}}} \mathbb{E}_{\mathbb{Q}}[X] = \Phi_0(X).$$

This being a contradiction, $\mathcal{Q}$ must be unique.

□

4 Convolution

In this section we study an operation on MCUFs called *convolution*. We know from the preceding section that an MCUF models the preferences of an agent. If this agent gets the possibility to trade in some financial market, this will affect her preference ordering. We shall see that this can be captured by convoluting appropriate MCUFs. From a purely mathematical point of view, the convolution is an operation on two MCUFs at time t which defines a new MCUF. If Φ^1 and Φ^2 are two (strongly) time-consistent DMCUFs, then we can obtain a new DMCUF by convoluting Φ^1_t and Φ^2_t at each time t. An important property of the convolution is that this DMCUF is again (strongly) time-consistent.

Definition 4.1 Let Φ^1_t and Φ^2_t be two MCUFs at time t. The *convolution of* Φ^1_t *and* Φ^2_t is defined as

$$\Phi^1_t \square \Phi^2_t(X) := \operatorname*{ess\,sup}_{Y\in\mathbf{L}^\infty} \Big\{ \Phi^1_t(X+Y) + \Phi^2_t(-Y) \Big\} \quad \text{for all } X \in \mathbf{L}^\infty. \tag{4.1}$$

If $\mathcal{B} \subseteq \mathbf{L}^\infty$ is non-empty, convex and $\mathcal{F}_t$-regular, the *convolution* of Φ^1_t and $\mathcal{B}$ is defined as

$$\Phi^1_t \square \mathcal{B}(X) := \operatorname*{ess\,sup}_{Y\in-\mathcal{B}} \Phi^1_t(X+Y) \quad \text{for } X \in \mathbf{L}^\infty. \tag{4.2}$$

Remark 4.2 i) The convolution is obviously symmetric, i.e.,

$$\Phi^1_t \square \Phi^2_t(X) = \Phi^2_t \square \Phi^1_t(X) \quad \text{for all } X \in \mathbf{L}^\infty.$$

ii) Since $\mathbf{L}^\infty$ is a linear space, we could equivalently define the convolution by

$$\Phi^1_t \square \Phi^2_t(X) := \operatorname*{ess\,sup}_{Y\in\mathbf{L}^\infty} \Big\{ \Phi^1_t(X-Y) + \Phi^2_t(Y) \Big\}.$$

This looks more natural because of the analogy to classical convolution operations. We deliberately choose the formulation (4.1) because it will turn out to be more convenient for subsequent interpretations. $\Diamond$

For a brief overview of the development of this type of convolution, we should probably start with Rockafellar. In his book [Roc70], he studied the *infimal convolution* of two convex functions f and g, defined as

$$f\Box g(x) := \inf_{y\in I\!R}\{f(x-y)+g(y)\}. \tag{4.3}$$

The terminology arises from the obvious analogy to the formula for classical integral convolutions. The convolution (4.3) is dual to the operation of addition for convex functions in the sense that the convex conjugate of $f+g$ is equal to the convolution of the conjugates of f and of g. As a purely mathematical concept, the above convolution was introduced and studied by [Del00] for static and coherent risk measures; see also [Del05] for an economic interpretation. One motivation for studying $\Phi_t^1\Box\Phi_t^2$ comes from a problem of risk transfer between two agents with preferences given by Φ_t^1 and Φ_t^2; see Barrieu/El Karoui ([BEK04], [BEK05]). We will show below that convoluting Φ_0^1 and Φ_0^2 also corresponds to finding a Pareto-efficient exchange between two individuals with preferences Φ_0^1 and Φ_0^2. This has been pointed out to us by N. Touzi; see also [JST05].

The main result of this section is an extension of Theorem 3.6 in [BEK05] in several directions. We show that the convolution operation produces a new MCUF and also preserves the dynamic property of (strong) time-consistency. All this is done in a conditional and abstract setting. This is in contrast to [BEK05] who only treat the static abstract case, and also to [BEK04] who study in the dynamic case a class of DMCUFs defined via BSDEs; we will come back to this in Section 7.1. Moreover, the question of time-consistency for convolutions of DMCUFs seems not to have been addressed so far in a general setting. In technical terms, the main difficulty here is like in section 2 related to closure properties of acceptance sets; this comes up when we need to identify the acceptance set of the convolution $\Phi_t^1\Box\Phi_t^2$. Before we state the main result of this section, we recall from Lemma 3.14 that any MCUF which is continuous from below is also continuous from above, and hence representable due to Theorem 3.19.

Theorem 4.3 *For $i=1,2$, let Φ_t^i be MCUFs at time t with acceptance sets $\mathcal{A}_t^i$ and concave conjugates α_t^i. Assume that $\Phi_t^1\Box\Phi_t^2(0)\in\mathbf{L}^\infty$. Then:*

a) *$\Phi_t^1\Box\Phi_t^2$ is an MCUF at time t, and*

$$\Phi_t^1\Box\Phi_t^2(X)=\Phi_t^1\Box\mathcal{A}_t^2(X)=\operatorname*{ess\,sup}_{Y\in-\mathcal{B}}\left\{\Phi_t^1(X+Y)+\Phi_t^2(-Y)\right\}\quad\text{for } X\in\mathbf{L}^\infty, \tag{4.4}$$

where $\mathcal{B}$ is an arbitrary subset of $\mathbf{L}^\infty$ containing $\mathcal{A}_t^2$.

b) *If Φ_t^1 and Φ_t^2 are both coherent, so is $\Phi_t^1\Box\Phi_t^2$.*

c) *If Φ_t^1 is continuous from below, then $\Phi_t^1\Box\Phi_t^2$ is continuous from below and in particular representable. Its concave conjugate $\alpha_t^{1\Box 2}$ is given by*

$$\alpha_t^{1\Box 2}(\mathbb{Q})=\alpha_t^1(\mathbb{Q})+\alpha_t^2(\mathbb{Q})\quad\text{for } \mathbb{Q}\in\mathcal{P}_t^{\approx}, \tag{4.5}$$

and its acceptance set $\mathcal{A}_t^{1\Box 2}$ is given by

$$\mathcal{A}_t^{1\Box 2}=\overline{\mathcal{A}_t^1+\mathcal{A}_t^2}, \tag{4.6}$$

where the closure is taken in $\sigma(\mathbf{L}^\infty, L^1)$. *If in addition we have*

$$\inf_{X \in \mathcal{A}_t^1 + \mathcal{A}_t^2} \mathbb{E}_{\tilde{\mathbb{Q}}}[X] > -\infty \quad \text{for some } \tilde{\mathbb{Q}} \in \mathcal{M}_1^e(\mathbb{P}), \tag{4.7}$$

then $\Phi_t^1 \Box \Phi_t^2$ *is also well-representable.*

d) *Suppose that* $\Phi^i = (\Phi_t^i)_{0 \le t \le T}$ *for* $i = 1, 2$ *are (strongly) time-consistent DMCUFs such that for each* $t \in [0, T]$, Φ_t^1 *is continuous from below and* $\Phi_t^1 \Box \Phi_t^2(0) \in \mathbf{L}^\infty$. *Then* $\Phi^1 \Box \Phi^2 = (\Phi_t^1 \Box \Phi_t^2)_{0 \le t \le T}$ *is also a (strongly) time-consistent DMCUF.*

Remark 4.4 i) Like in section 3, condition (4.7) simplifies if $\Phi^1 \Box \Phi^2$ is strongly time-consistent; it is then enough if $\inf_{X \in \mathcal{A}_0^1 + \mathcal{A}_0^2} \mathbb{E}_{\tilde{\mathbb{Q}}}[X] = \alpha_0^1(\tilde{\mathbb{Q}}) + \alpha_0^2(\tilde{\mathbb{Q}}) > -\infty$ for some $\tilde{\mathbb{Q}} \in \mathcal{M}_1^e(\mathbb{P})$.

ii) $\Phi_t^1 \Box \Phi_t^2$ need not be normalized even if Φ_t^1 and Φ_t^2 both are. This is our main reason for abandoning the requirement of normalization.

◊

As mentioned above, convoluting the MCUFs Φ_0^1 and Φ_0^2 corresponds to finding a Pareto-efficient exchange between two individuals with preferences corresponding to Φ_0^1 respectively Φ_0^2. To see this, denote by $K_0 := \left\{ (Y^1, Y^2) \in \mathbf{L}^\infty \times \mathbf{L}^\infty \,\middle|\, Y^1 + Y^2 = X \right\}$ the set of all *feasible exchanges*. Then (4.1) for $t = 0$ can equivalently be written as

$$\sup_{(Y^1, Y^2) \in K_0} \left\{ \Phi_0^1(Y^1) + \Phi_0(Y^2) \right\}. \tag{4.8}$$

A feasible exchange $(\hat{Y}^1, \hat{Y}^2) \in K_0$ is called *Pareto-efficient* if there exists no $(Y^1, Y^2) \in K_0$ such that

$$\Phi_0^i(Y^i) \ge \Phi_0^i(\hat{Y}^i) \quad \text{and} \quad \Phi_0^j(Y^j) > \Phi_0^j(\hat{Y}^j) \quad \text{for } (i, j) = (1, 2) \text{ or } (i, j) = (2, 1). \tag{4.9}$$

$(\hat{Y}^1, \hat{Y}^2)$ is called *weakly Pareto-efficient* if "$\ge$" is replaced by "$>$" in (4.9). It is well known (see, e.g., Proposition 2.8 in [IBK02]) that $(\hat{Y}^1, \hat{Y}^2) \in K_0$ is weakly Pareto-efficient if and only if there exists $(\lambda^1, \lambda^2) \in I\!R_+^2 \backslash \{(0, 0)\}$ such that

$$(\hat{Y}^1, \hat{Y}^2) \quad \text{maximizes} \quad (Y^1, Y^2) \mapsto \lambda^1 \Phi_0^1(Y^1) + \lambda^2 \Phi_0^2(Y^2) \quad \text{over } K_0. \tag{4.10}$$

If $\lambda^1 > 0$ and $\lambda^2 > 0$ then $(\hat{Y}^1, \hat{Y}^2)$ is even Pareto-efficient.

Note that for any $c \in I\!R$ and $(Y^1, Y^2) \in K_0$ also $(Y^1 + c, Y^2 - c) \in K_0$ and that by translation invariance of Φ_0^i, we have

$$\lambda^1 \, \Phi_0^1(Y^1 + c) + \lambda^2 \, \Phi_0^2(Y^2 - c) = \lambda^1 \, \Phi_0^1(Y^1) + \lambda^2 \, \Phi_0^2(Y^2) + c(\lambda^1 - \lambda^2).$$

But, if $\lambda^1 \neq \lambda^2$ then this tends to $+\infty$ if $c \to +\infty$ or if $c \to -\infty$. Thus $(\hat{Y}^1, \hat{Y}^2) \in K_0$ is a Pareto-efficient exchange if and only if it satisfies (4.10) for $\lambda^1 = \lambda^2 > 0$, i.e., if it maximizes (4.8). This explains the connection between the convolution and Pareto-efficient exchanges.

There is another economic interpretation for the convolution which comes from the second expression in (4.4) and was suggested in [BEK05]. Consider two individuals I_1 and I_2 with preferences corresponding to Φ_t^1 and Φ_t^2 who want to maximize their monetary utilities. Suppose that I_1 owns at time $t < T$ some asset with payoff X at time T. She might try to increase her utility by exchanging at time t with I_2 some payoff Y due at time T. But of course, I_2 will only agree to hand over Y to I_1 if he deems the for him resulting payoff $-Y$ acceptable. This gives a constraint for the maximization problem of I_1 exactly as in (4.4). In particular, if the preferences of agent I_2 correspond to a normalized MCUF so that $\Phi_t^2(0) = 0$, he will agree to handing over Y if and only if this does not decrease his utility.

In the proof of Theorem 4.3, we use the following auxiliary result.

Lemma 4.5 *Take an MCUF Φ_t^1 at time t and a non-empty, convex and $\mathcal{F}_t$-regular set $\mathcal{B} \subseteq \mathbf{L}^\infty$. If $\Phi_t^1 \Box \mathcal{B}(0) \in \mathbf{L}^\infty$, then:*

a) *$\Phi_t^1 \Box \mathcal{B}$ is an MCUF at time t.*
b) *If Φ_t^1 is coherent and $\mathcal{B}$ a convex cone containing 0, then $\Phi_t^1 \Box \mathcal{B}$ is an MCohUF at time t.*
c) *If Φ_t^1 is continuous from below, so is $\Phi_t^1 \Box \mathcal{B}$.*

Proof To shorten notation we write $\Phi_t := \Phi_t^1 \Box \mathcal{B}$.

a) Properties A) and B) of Definition 3.1 are obvious. To see concavity, let $X_1, X_2 \in \mathbf{L}^\infty$ and $\beta \in [0, 1]$. Since $\mathcal{B}$ is convex and Φ_t^1 is concave, we get

$$\begin{aligned}
&\Phi_t(\beta X_1 + (1-\beta)X_2) \\
&\quad = \operatorname*{ess\,sup}_{Y_1, Y_2 \in -\mathcal{B}} \Phi_t^1\left(\beta(X_1 + Y_1) + (1-\beta)(X_2 + Y_2)\right) \\
&\quad \geq \beta \operatorname*{ess\,sup}_{Y_1 \in -\mathcal{B}} \Phi_t^1(X_1 + Y_1) + (1-\beta) \operatorname*{ess\,sup}_{Y_2 \in -\mathcal{B}} \Phi_t^1(X_2 + Y_2) \\
&\quad = \beta \Phi_t(X_1) + (1-\beta)\Phi_t(X_2).
\end{aligned}$$

Finally, A) and B) imply

$$\|\Phi_t(X)\|_{\mathbf{L}^\infty} \leq \|\Phi_t(0)\|_{\mathbf{L}^\infty} + \|X\|_{\mathbf{L}^\infty} < \infty$$

so that $\Phi_t(X) \in \mathbf{L}^\infty$ for each $X \in \mathbf{L}^\infty$.

b) To see that Φ_t is coherent, we first show that

$$\Phi_t(0) = 0. \tag{4.11}$$

Suppose this is not true. Since Φ_t^1 as MCohUF is normalized so that $0 \in \mathcal{B}$ implies $\Phi_t(0) \geq \Phi_t^1(0 + 0) = 0$, we then must have $\Phi_t(0) > 0$ with positive probability. Because the essential supremum in the definition of Φ_t can be written as the pointwise supremum over a countable number of elements of $-\mathcal{B}$ ([Nev75]), there exist $\overline{Y} \in -\mathcal{B}$ and $A \in \mathcal{F}_t$ with $\mathbb{P}[A] > 0$ such that

$$\Phi_t(0) \geq \Phi_t^1(\overline{Y}) > 0 \quad \text{on } A.$$

Now replace $\overline{Y}$ by $n\overline{Y}$ and use positive homogeneity of Φ_t^1 and that $\mathcal{B}$ is a convex cone to obtain for $n \to \infty$ that $\Phi_t(0) = +\infty$ on A, contradicting $\Phi_t(0) \in \mathbf{L}^\infty$. This establishes (4.11). Now let $\lambda > 0$. For any $X \in \mathbf{L}^\infty$ we obtain by using positive homogeneity of Φ_t^1 and the fact that $\mathcal{B}$ is a convex cone that

$$\begin{aligned}
\operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \Phi_t^1(\lambda X + Y) &= \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \left\{ \lambda \Phi_t^1\Big(X + \frac{Y}{\lambda}\Big)\Big) \right\} \\
&= \lambda \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \Phi_t^1(X + Y).
\end{aligned}$$

Hence Φ_t is positively homogeneous.

c) To see that continuity from below of Φ_t^1 carries over to Φ_t, let $(X_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence increasing to some $X \in \mathbf{L}^\infty$. Then monotonicity of Φ_t yields

$$\begin{aligned}\lim_{n\to\infty} \Phi_t(X_n) &= \sup_{n\in\mathbb{N}} \Phi_t(X_n) \\ &= \sup_{n\in\mathbb{N}} \left\{ \operatorname*{ess\,sup}_{Y\in-\mathcal{B}} \Phi_t^1(X_n+Y)\right\} \\ &= \operatorname*{ess\,sup}_{Y\in-\mathcal{B}} \left\{ \sup_{n\in\mathbb{N}} \Phi_t^1(X_n+Y)\right\} \\ &= \operatorname*{ess\,sup}_{Y\in-\mathcal{B}} \Phi_t^1(X+Y) \\ &= \Phi_t(X),\end{aligned}$$

which shows that Φ_t is continuous from below. $\square$

Proof of Theorem 4.3 To shorten notation we write $\Phi_t := \Phi_t^1 \Box \Phi_t^2$ for $t \in [0,T]$.

a) Once we have shown (4.4), the rest follows from Lemma 4.5 a). We begin by proving the first equality in (4.4), i.e., that

$$\operatorname*{ess\,sup}_{Y\in\mathbf{L}^\infty} \left\{\Phi_t^1(X+Y)+\Phi_t^2(-Y)\right\} = \operatorname*{ess\,sup}_{Y'\in-\mathcal{A}_t^2} \Phi_t^1(X+Y').$$

For arguing "$\leq$", we fix $Y \in \mathbf{L}^\infty$ and show that there exists $Y' \in -\mathcal{A}_t^2$ such that

$$\Phi_t^1(X+Y)+\Phi_t^2(-Y) = \Phi_t^1(X+Y').$$

In fact, translation invariance implies that $Y' := Y + \Phi_t^2(-Y)$ is in $-\mathcal{A}_t^2$ and also yields $\Phi_t^1(X+Y') = \Phi_t^1(X+Y)+\Phi_t^2(-Y)$. To see "$\geq$", note that $Y' \in -\mathcal{A}_t^2$ yields $\Phi_t^2(-Y') \geq 0$ and therefore

$$\Phi_t^1(X+Y') \leq \Phi_t^1(X+Y')+\Phi_t^2(-Y').$$

This shows the first equality in (4.4) which then immediately implies the second by

$$\begin{aligned}\Phi_t^1\Box\Phi_t^2(X) &\geq \operatorname*{ess\,sup}_{Y'\in-\mathcal{B}} \left\{\Phi_t^1(X+Y')+\Phi_t^2(-Y')\right\} \\ &\geq \operatorname*{ess\,sup}_{Y'\in-\mathcal{A}_t^2} \left\{\Phi_t^1(X+Y')+\Phi_t^2(-Y')\right\} \\ &\geq \operatorname*{ess\,sup}_{Y'\in-\mathcal{A}_t^2} \Phi_t^1(X+Y') \\ &= \Phi_t^1\Box\Phi_t^2(X),\end{aligned}$$

where we used again that $\Phi_t^2(-Y') \geq 0$ for all $Y' \in -\mathcal{A}_t^2$.

b) This follows immediately from (4.4) and Lemma 4.5 b), since $\mathcal{A}_t^2$ is by Lemma 3.6 a convex cone containing 0.

c) Continuity from below follows immediately from (4.4) and Lemma 4.5. From this together with a) and Lemma 3.14, we can apply Theorem 3.19 which implies that Φ_t

is representable. If in addition (4.7) holds, Φ_t is even well-representable by Theorem 3.16. Moreover, (4.5) holds since by Definition 3.11 for any $\mathbb{Q} \in \mathcal{P}_t^{\approx}$

$$\begin{aligned}
\alpha_t^{1\square 2}(\mathbb{Q}) &= \operatorname*{ess\,inf}_{X\in\mathbf{L}^\infty}\Big\{\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \operatorname*{ess\,sup}_{Y\in\mathbf{L}^\infty}\Big\{\Phi_t^1(X+Y)+\Phi_t^2(-Y)\Big\}\Big\} \\
&= \operatorname*{ess\,inf}_{X\in\mathbf{L}^\infty} \\
&\quad \Big\{\operatorname*{ess\,inf}_{Y\in\mathbf{L}^\infty}\Big\{\mathbb{E}_{\mathbb{Q}}[X+Y|\mathcal{F}_t]+\mathbb{E}_{\mathbb{Q}}[-Y|\mathcal{F}_t]-\Phi_t^1(X+Y)-\Phi_t^2(-Y)\Big\}\Big\} \\
&= \operatorname*{ess\,inf}_{Y\in\mathbf{L}^\infty}\Big\{\mathbb{E}_{\mathbb{Q}}[-Y|\mathcal{F}_t]-\Phi_t^2(-Y) \\
&\qquad\qquad + \operatorname*{ess\,inf}_{X\in\mathbf{L}^\infty}\Big\{\mathbb{E}_{\mathbb{Q}}[X+Y|\mathcal{F}_t]-\Phi_t^1(X+Y)\Big\}\Big\} \\
&= \operatorname*{ess\,inf}_{Y\in\mathbf{L}^\infty}\Big\{\mathbb{E}_{\mathbb{Q}}[-Y|\mathcal{F}_t]-\Phi_t^2(-Y)+\alpha_t^1(\mathbb{Q})\Big\} \\
&= \alpha_t^2(\mathbb{Q})+\alpha_t^1(\mathbb{Q}).
\end{aligned}$$

The proof of the assertion that $\mathcal{A}_t^{1\square 2} = \overline{\mathcal{A}_t^1+\mathcal{A}_t^2}$ is a bit more involved. If $X_i \in \mathcal{A}_t^i$ for $i=1,2$, then $\Phi_t(X_1+X_2) \ge \Phi_t^1(X_1)+\Phi_t^2(X_2) \ge 0$ shows that $X_1+X_2 \in \mathcal{A}_t^{1\square 2}$, and because $\mathcal{A}_t^{1\square 2}$ is closed in $\sigma(\mathbf{L}^\infty, L^1)$ by Theorem 3.19, we obtain $\overline{\mathcal{A}_t^1+\mathcal{A}_t^2} \subseteq \mathcal{A}_t^{1\square 2}$. For the converse inclusion, we claim that

$$\inf_{X\in\mathcal{A}_t^{1\square 2}}\mathbb{E}[ZX] = \inf_{X\in\mathcal{A}_t^1+\mathcal{A}_t^2}\mathbb{E}[ZX] = \inf_{X\in\overline{\mathcal{A}_t^1+\mathcal{A}_t^2}}\mathbb{E}[ZX] \quad \text{for all } Z\in L_+^1; \tag{4.12}$$

note that the second equality follows from the first since we already know that $\mathcal{A}_t^1+\mathcal{A}_t^2 \subseteq \overline{\mathcal{A}_t^1+\mathcal{A}_t^2} \subseteq \mathcal{A}_t^{1\square 2}$. Then if the inclusion "$\subseteq$" in (4.6) is not true, there exists some $X' \in \mathcal{A}_t^{1\square 2}\setminus\overline{\mathcal{A}_t^1+\mathcal{A}_t^2}$, and the Hahn-Banach theorem yields some $Z' \in L^1$ with

$$\inf_{X\in\overline{\mathcal{A}_t^1+\mathcal{A}_t^2}}\mathbb{E}[XZ'] > \mathbb{E}[X'Z'] > -\infty. \tag{4.13}$$

But since $-\big(\overline{\mathcal{A}_t^1+\mathcal{A}_t^2}\big)$ is solid, we must have $Z' \ge 0$, and so (4.13) contradicts (4.12).

To complete the proof, it remains to establish (4.12). To that end, we first use Lemma 3.12, (4.5) and again Lemma 3.12 to obtain

$$\begin{aligned}
\operatorname*{ess\,inf}_{X\in\mathcal{A}_t^{1\square 2}}\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] &= \alpha_t^{1\square 2}(\mathbb{Q}) \\
&= \operatorname*{ess\,inf}_{X_1\in\mathcal{A}_t^1}\mathbb{E}_{\mathbb{Q}}[X_1|\mathcal{F}_t]+\operatorname*{ess\,inf}_{X_2\in\mathcal{A}_t^2}\mathbb{E}_{\mathbb{Q}}[X_2|\mathcal{F}_t] \\
&= \operatorname*{ess\,inf}_{X\in\mathcal{A}_t^1+\mathcal{A}_t^2}\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] \quad \text{for all } \mathbb{Q}\in\mathcal{P}_t^{\approx}.
\end{aligned} \tag{4.14}$$

Now up to normalization, $\mathcal{P}_t^{\approx}$ can be identified with

$$\mathcal{Z}_t := \Big\{Z\in L_+^1 \;\Big|\; \text{for all } A\in\mathcal{F}_t,\ \mathbb{P}[A]=0 \text{ iff } Z\mathbf{1}_A=0\Big\}.$$

Hence (4.14) implies that

$$\operatorname*{ess\,inf}_{X\in\mathcal{A}_t^{1\square 2}}\mathbb{E}[ZX|\mathcal{F}_t] = \operatorname*{ess\,inf}_{X\in\mathcal{A}_t^1+\mathcal{A}_t^2}\mathbb{E}[ZX|\mathcal{F}_t] \quad \text{for all } Z\in\mathcal{Z}_t. \tag{4.15}$$

To extend this to all $Z \in L^1_+$, fix $Z \in L^1_+$ and define $B \in \mathcal{F}_t$ up to nullsets by $\mathbf{1}_B := \operatorname{ess\,sup}\{\mathbf{1}_A \,|\, A \in \mathcal{F}_t \text{ and } Z\mathbf{1}_A = 0\}$ so that $Z\mathbf{1}_{B^c} = Z$. Because Φ_t is representable, we have by Lemma 3.12 that

$$\mathbf{L}^\infty \ni -\Phi_t(0) = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{P}_t^=} \alpha_t^{1\Box 2}(\mathbb{Q}) = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{P}_t^=} \Big(\operatorname*{ess\,inf}_{X\in\mathcal{A}_t^{1\Box 2}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t]\Big)$$

and so there exists some $\mathbb{Q}' \in \mathcal{P}_t^=$ with density Z'_T such that $\operatorname*{ess\,inf}_{X\in\mathcal{A}_t^{1\Box 2}} \mathbb{E}_{\mathbb{Q}'}[X|\mathcal{F}_t] \in \mathbf{L}^\infty$. Then $\hat{Z} := Z'_T\mathbf{1}_B + Z\mathbf{1}_{B^c}$ is in $\mathcal{Z}_t$ and

$$\mathbf{1}_{B^c}\mathbb{E}[ZX|\mathcal{F}_t] = \mathbf{1}_{B^c}\mathbb{E}[\hat{Z}X|\mathcal{F}_t]. \tag{4.16}$$

Using $Z = Z\mathbf{1}_{B^c}$, (4.16), (4.15) for $\hat{Z}$ and then reversing the steps again yields

$$\operatorname*{ess\,inf}_{X\in\mathcal{A}_t^{1\Box 2}} \mathbb{E}[ZX|\mathcal{F}_t] = \operatorname*{ess\,inf}_{X\in\mathcal{A}_t^1+\mathcal{A}_t^2} \mathbb{E}[ZX|\mathcal{F}_t]$$

as desired. Because $\big\{\mathbb{E}[ZX|\mathcal{F}_t] \,\big|\, X \in \mathcal{B}\big\}$ is a lattice for $\mathcal{B} \in \big\{\mathcal{A}_t^{1\Box 2}, \mathcal{A}_t^1 + \mathcal{A}_t^2\big\}$ by $\mathcal{F}_t$-regularity, we can interchange infimum and expectation to obtain

$$\inf_{X\in\mathcal{A}_t^{1\Box 2}} \mathbb{E}[ZX] = \inf_{X\in\mathcal{A}_t^1+\mathcal{A}_t^2} \mathbb{E}[ZX]$$

for every $Z \in L^1_+$. This establishes (4.12).

d) Suppose first that Φ^1 and Φ^2 are time-consistent. We may also assume that they are normalized, because the MCUFs $\hat{\Phi}^i_u(X) := \Phi^i_u(X) - \Phi^i_u(0)$ for $i = 1, 2$ are, we have $\Phi_u(X) = \hat{\Phi}^1_u\Box\hat{\Phi}^2_u(X) + \big(\Phi^1_u(0) + \Phi^2_u(0)\big)$, and time-consistency is not affected by translation. So let $s \leq t$ and X_1, X_2 be such that

$$\Phi_t(X_1) = \Phi_t(X_2) = \operatorname*{ess\,sup}_{Y\in-\mathcal{A}_t^2} \Phi^1_t(X_2 + Y). \tag{4.17}$$

By (4.4) it suffices to show that we then have

$$\Phi^1_s\Box\mathcal{A}^2_s(X_1) = \operatorname*{ess\,sup}_{Y'\in-\mathcal{A}_s^2} \Phi^1_s(X_1 + Y') = \operatorname*{ess\,sup}_{Y'\in-\mathcal{A}_s^2} \Phi^1_s(X_2 + Y').$$

Now Lemma 3.25 implies that

$$\begin{aligned} \Phi^1_s(X) &= \Phi^1_s\big(\Phi^1_t(X)\big) \quad \text{for } X \in \mathbf{L}^\infty, & (4.18)\\ \mathcal{A}^2_s &= \mathcal{A}^2_s(\mathcal{F}_t) + \mathcal{A}^2_t, & (4.19) \end{aligned}$$

and Lemma 3.6 applied to $\mathcal{A}^2_t$ and $\mathcal{F}_t$-regularity of Φ^1_t yield that $\big\{\Phi^1_t(X+Y) \,\big|\, Y \in -\mathcal{A}^2_t\big\}$ is a lattice for any $X \in \mathbf{L}^\infty$. Hence there is a sequence (Y_n) in $-\mathcal{A}^2_t$ such that $\operatorname*{ess\,sup}_{Y\in-\mathcal{A}_t^2} \Phi^1_t(X+Y) = \nearrow -\lim_{n\to\infty} \Phi^1_t(X + Y_n)$. Moreover, $\big(\Phi^1_t(X + Y_n)\big)_{n\in\mathbb{N}}$ is uniformly bounded due to (4.4) because

$$-\|X + Y_1\|_{\mathbf{L}^\infty} \leq \Phi^1_t(X + Y_1) \leq \Phi^1_t(X + Y_n) \leq \operatorname*{ess\,sup}_{Y\in-\mathcal{A}_t^2} \Phi^1_t(X + Y) = \Phi_t(X) \in \mathbf{L}^\infty.$$

Hence translation invariance and continuity from below of Φ^1_s imply for any $\hat{Y} \in \mathcal{A}^2_s(\mathcal{F}_t)$ that

$$\Phi^1_s\left(\operatorname*{ess\,sup}_{Y\in-\mathcal{A}^2_t}\Phi^1_t(X+Y+\hat{Y})\right) = \nearrow -\lim_{n\to\infty}\Phi^1_s\big(\Phi^1_t(X+Y_n)+\hat{Y}\big)$$
$$\leq \operatorname*{ess\,sup}_{Y\in-\mathcal{A}^2_t}\Phi^1_s\big(\Phi^1_t(X+Y)+\hat{Y}\big),$$

and by monotonicity of Φ^1_s, we even must have equality. Combining this with (4.18), (4.19) and using (4.17) to exchange X_1 for X_2, we get

$$\begin{aligned}\operatorname*{ess\,sup}_{Y'\in-\mathcal{A}^2_s}\Phi^1_s(X_1+Y') &= \operatorname*{ess\,sup}_{\hat{Y}\in-\mathcal{A}^2_s(\mathcal{F}_t)}\ \operatorname*{ess\,sup}_{Y\in-\mathcal{A}^2_t}\Phi^1_s\big(\Phi^1_t(X_1+Y+\hat{Y})\big)\\ &= \operatorname*{ess\,sup}_{\hat{Y}\in-\mathcal{A}^2_s(\mathcal{F}_t)}\ \Phi^1_s\left(\operatorname*{ess\,sup}_{Y\in-\mathcal{A}^2_t}\Phi^1_t(X_2+Y)+\hat{Y}\right)\\ &= \operatorname*{ess\,sup}_{Y'\in-\mathcal{A}^2_s}\ \Phi^1_s(X_2+Y'),\end{aligned}$$

where the last equality is obtained by doing the same steps in reverse order with X_1 replaced by X_2. This shows that Φ is time-consistent. If Φ^1, Φ^2 are strongly time-consistent, we have in addition $\mathcal{A}^i_t \subseteq \mathcal{A}^i_s$ for $t \geq s$ and $i = 1, 2$, and thus also $\overline{\mathcal{A}^1_t + \mathcal{A}^2_t} \subseteq \overline{\mathcal{A}^1_s + \mathcal{A}^2_s}$. Hence (4.6) implies that Φ is strongly time-consistent as well, and so d) is proved.

□

Remark 4.6 Parts of the proof of Theorem 4.3 are a straightforward generalization of the arguments for the (static) Theorem 3.6 in [BEK05]; this extends smoothly because thanks to the preparations in section 3, we can appeal to the dynamic representations in Theorem 3.16 and 3.19 instead of its static counterpart. Exceptions are the parts where we show (4.6) and the assertions b) and d). ◊

If Φ^1_t is an MCUF and $\mathcal{B}$ an acceptable set at time t, Lemma 4.5 implies that $\Phi_t := \Phi^1_t \Box \mathcal{B}$ is again an MCUF, provided that $\Phi_t(0) \in \mathbf{L}^\infty$. In the sequel, we want to have a maximum of good properties for that Φ_t with a minimum of assumptions on $\mathcal{B}$. To make this more precise, recall from Lemma 3.8 the MCUF $\Phi^{\mathcal{B}}_t$ associated to $\mathcal{B}$. By (4.4), it seems natural to expect that $\Phi^1_t \Box \mathcal{B} = \Phi^1_t \Box \Phi^{\mathcal{B}}_t$ and that the acceptance set of Φ_t should be $\overline{\mathcal{A}^1_t + \mathcal{B}}$ in view of (4.6). However, this can be deduced from the preceding results only if Φ^1_t is continuous from below and $\mathcal{B}$ is the acceptance set of $\Phi^{\mathcal{B}}_t$, e.g., if $\mathcal{B}$ is closed in $\sigma(\mathbf{L}^\infty, L^1)$. Because the latter is often hard to check, we do not want to make that assumption. So we first work with the $\sigma(\mathbf{L}^\infty, L^1)$-closure $\overline{\mathcal{B}}$ of $\mathcal{B}$ since we have precise results for $\Phi^1_t \Box \Phi^{\overline{\mathcal{B}}}_t$, and then show that the latter coincides with $\Phi^1_t \Box \mathcal{B}$.

The program sketched above is carried out in the next result. This in turn is used below in section 6 when we study utility indifference valuation.

Proposition 4.7 *Let $\mathcal{B}$ be an acceptable set and Φ^1_t an MCUF at time t with acceptance set $\mathcal{A}^1_t$ and concave conjugate α^1_t. Denote by $\overline{\mathcal{B}}$ the closure of $\mathcal{B}$ in $\sigma(\mathbf{L}^\infty, L^1)$. If*

$\Phi_t^1 \Box \mathcal{B}(0) = \operatorname{ess\,sup}_{Y \in -\mathcal{B}} \Phi_t^1(Y) \in \mathbf{L}^\infty$, *then*

$$\Phi_t^1 \Box \mathcal{B} = \Phi_t^1 \Box \Phi_t^{\mathcal{B}}. \tag{4.20}$$

If in addition Φ_t^1 *is continuous from below and*

$$\operatorname{ess\,sup} \big(-\overline{\mathcal{B}} \cap \mathbf{L}^\infty(\mathcal{F}_t) \big) \in \mathbf{L}^\infty, \tag{4.21}$$

then

$$\Phi_t^1 \Box \mathcal{B} = \Phi_t^1 \Box \Phi_t^{\overline{\mathcal{B}}}. \tag{4.22}$$

In particular, $\Phi_t := \Phi_t^1 \Box \mathcal{B}$ *is then continuous from below, with concave conjugate*

$$\alpha_t(\mathbb{Q}) = \alpha_t^1(\mathbb{Q}) + \alpha_t^{\overline{\mathcal{B}}}(\mathbb{Q}) := \alpha_t^1(\mathbb{Q}) + \operatorname*{ess\,inf}_{Y \in \overline{\mathcal{B}}} \mathbb{E}_{\mathbb{Q}}[Y | \mathcal{F}_t] \tag{4.23}$$

and acceptance set

$$\mathcal{A}_t = \overline{\mathcal{A}_t^1 + \overline{\mathcal{B}}} = \overline{\mathcal{A}_t^1 + \mathcal{B}}.$$

Proof If $\mathcal{A}_t^{\mathcal{B}}$ denotes the acceptance set of $\Phi_t^{\mathcal{B}}$, then $\mathcal{B} \subseteq \mathcal{A}_t^{\mathcal{B}}$ so that (4.4) implies

$$\Phi_t^1 \Box \Phi_t^{\mathcal{B}}(X) = \operatorname*{ess\,sup}_{Y \in -\mathcal{A}_t^{\mathcal{B}}} \Phi_t^1(X+Y) \geq \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \Phi_t^1(X+Y) = \Phi_t^1 \Box \mathcal{B}(X). \tag{4.24}$$

Since $\Phi_t^{\mathcal{B}}$ is non-negative on $\mathcal{A}_t^{\mathcal{B}}$, (4.4) also yields

$$\begin{aligned} \Phi_t^1 \Box \Phi_t^{\mathcal{B}}(X) &\leq \operatorname*{ess\,sup}_{Y \in -\mathcal{A}_t^{\mathcal{B}}} \big(\Phi_t^1(X+Y) + \Phi_t^{\mathcal{B}}(-Y) \big) \\ &\leq \operatorname*{ess\,sup}_{Y \in \mathbf{L}^\infty} \big(\Phi_t^1(X+Y) + \Phi_t^{\mathcal{B}}(-Y) \big) \\ &= \Phi_t^1 \Box \Phi_t^{\mathcal{B}}(X) \end{aligned}$$

so that $\Phi_t^1 \Box \Phi_t^{\mathcal{B}}(X) = \operatorname*{ess\,sup}_{Y \in -\mathcal{A}_t^{\mathcal{B}}} \big(\Phi_t^1(X+Y) + \Phi_t^{\mathcal{B}}(-Y) \big)$. In view of (4.24), it thus suffices to show that for each $Y' \in -\mathcal{A}_t^{\mathcal{B}}$,

$$\Phi_t^1(X+Y') + \Phi_t^{\mathcal{B}}(-Y') \leq \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \Phi_t^1(X+Y). \tag{4.25}$$

Pick a sequence (m_t^n) in $\mathbf{L}^\infty(\mathcal{F}_t)$ and an $\mathcal{F}_t$-partition (A_n) with $-Y' - m_t^n \in \mathcal{B}$ for all n and

$$\Phi_t^{\mathcal{B}}(-Y') \leq \sum_{n=1}^{\infty} \mathbf{1}_{A_n} m_t^n + \epsilon,$$

for a fixed $\epsilon > 0$. Then translation invariance of Φ_t^1 implies that

$$\begin{aligned}\operatorname*{ess\,sup}_{Y\in-\mathcal{B}} \Phi_t^1(X+Y) &= \sum_{n=1}^{\infty} \mathbf{1}_{A_n} \operatorname*{ess\,sup}_{Y\in-\mathcal{B}} \Phi_t^1(X+Y) \\ &\geq \sum_{n=1}^{\infty} \mathbf{1}_{A_n} \Phi_t^1(X+Y'+m_t^n) \\ &= \Phi_t^1(X+Y') + \sum_{n=1}^{\infty} \mathbf{1}_{A_n} m_t^n \\ &\geq \Phi_t^1(X+Y') + \Phi_t^{\mathcal{B}}(-Y') - \epsilon.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, this proves (4.25) and hence (4.20).

If we now assume (4.21), $\overline{\mathcal{B}}$ is like $\mathcal{B}$ acceptable at time t and thus by Lemma 3.8 the acceptance set of the MCUF $\Phi_t^{\overline{\mathcal{B}}}$. So it is enough to prove (4.22) because all claimed properties then follow from Theorem 4.3 and Lemma 3.12, and as $\Phi_t := \Phi_t^1 \Box \mathcal{B}$ and $\Phi_t^1 \Box \Phi_t^{\overline{\mathcal{B}}}$ both are MCUFs at time t, they coincide if their acceptance sets $\mathcal{A}_t$ and $\overline{\mathcal{A}_t^1 + \overline{\mathcal{B}}} = \overline{\mathcal{A}_t^1 + \mathcal{B}}$ agree. By the assumptions and Lemma 4.5, Φ_t is continuous from below, so that $\mathcal{A}_t$ is closed in $\sigma(\mathbf{L}^\infty, L^1)$ by Lemma 3.14 and Theorem 3.19. Because the definition of Φ_t gives $\mathcal{A}_t^1 + \mathcal{B} \subseteq \mathcal{A}_t$, we obtain $\overline{\mathcal{A}_t^1 + \mathcal{B}} \subseteq \mathcal{A}_t$, and the converse inclusion is trivial since (4.2) and (4.4) with $\mathcal{A}_t^2 = \overline{\mathcal{B}}$ give

$$\Phi_t(X) \leq \operatorname*{ess\,sup}_{Y\in-\overline{\mathcal{B}}} \Phi_t^1(X+Y) = \Phi_t^1 \Box \Phi_t^{\overline{\mathcal{B}}}(X) \quad \text{for } X \in \mathbf{L}^\infty.$$

This completes the proof. □

5 Superhedging under Constraints

This section deals with superhedging under constraints. The results presented here are slight modifications of those Föllmer and Kramkov proved in [FK97]. We obtain the existence of a minimal hedging portfolio for a given payoff if trading is constrained, and we provide a representation of the value process corresponding to this portfolio. These results will be very helpful in section 6. There we consider a DMCUF Φ representing the preferences of some agent and assume that she gets the possibility to trade in a financial market, possibly under some constraints. Then we use the value process of the minimal hedging portfolio to construct a strongly time-consistent DMCUF which allows us to capture the effects on the agent's preference order of the trading opportunities.

In this section, all processes (except for integrands of stochastic integrals) are assumed to be RCLL and adapted with respect to the given filtration $\mathbb{F}$. For two such processes U and V, the relation $U \preceq V$ means that $V - U$ is an increasing process. We model the discounted price process of some traded assets by a locally bounded $\mathbb{R}^d$-valued $\mathbb{P}$-semimartingale $S = (S_t)_{0\leq t\leq T}$. Before we can state the main theorem of this section, we need to specify the set of strategies allowed for trading and provide some technical results which are required for its proof.

Definition 5.1 We denote by $L(S)$ the set of all $\mathbb{R}^d$-valued predictable processes $H = (H_t)_{0\le t\le T}$ which are S-integrable, and call $H \in L(S)$ an *admissible strategy* if the process $(\int_0^t H_u\, dS_u)_{0\le t\le T}$ is locally bounded from below. The set of all admissible strategies is denoted by $L^a_{\text{loc}}(S)$. We call a triple (x, H, K) an *admissible portfolio* if $x \in \mathbb{R}$, $H \in L^a_{\text{loc}}(S)$ and $K = (K_t)_{0\le t\le T}$ is an adapted RCLL increasing process with $K_0 = 0$. The corresponding *value process* is defined by

$$V_t = x + \int_0^t H_s\, dS_s - K_t, \quad t \in [0, T].$$

The economic interpretation of an admissible portfolio (x, H, K) is very simple: x gives the initial capital of the portfolio, H specifies the number of units of each asset held in the portfolio, and K models cumulative consumption.

If trading is not constrained, every admissible strategy can be used for trading. However, we want to allow for trading constraints. For technical reasons we need to impose some closedness properties on the set of allowed hedging strategies. To that end, we recall the *Émery distance* between two real-valued semimartingales N^1 and N^2, defined as

$$D(N^1, N^2) = \sup_{|J|\le 1} \mathbb{E}\left[1 \wedge \int_0^T J_s\, d(N^1 - N^2)_s\right],$$

where the supremum is taken over all predictable processes J which are uniformly bounded by 1. By Theorem 5.4 of [Mem80], the space $L(S)$ is complete with respect to the metric

$$d_S(H, G) = D\left(\int H\, dS, \int G\, dS\right).$$

Definition 5.2 We call a subset $\mathcal{H}$ of $L^a_{\text{loc}}(S)$ an *admissible hedging set* if it contains $H \equiv 0$, is closed in $L^a_{\text{loc}}(S)$ with respect to the metric d_S and is *predictably convex*, i.e., for any $H, G \in \mathcal{H}$ and any $[0,1]$-valued predictable process $h = (h_t)_{0\le t\le T}$, the process $hH + (1-h)G$ belongs to $\mathcal{H}$. An admissible portfolio (x, H, K) is called $\mathcal{H}$*-constrained* if $H \in \mathcal{H}$.

Remark 5.3 i) Note that $\mathcal{H}$ need not be closed under addition or multiplication by scalars in general.

ii) Since $\mathcal{H}$ is predictably convex and contains 0, we have for every $H \in \mathcal{H}$ and every stopping time τ that also $H' := H\mathbf{1}_{]\!]\tau,T]\!]} \in \mathcal{H}$. Note that for any $H \in L^a_{\text{loc}}(S)$, such a H' is also in $L^a_{\text{loc}}(S)$. More generally, if N is any process which is locally bounded from below and τ is any stopping time, then $N' := N - N^\tau$ is again locally bounded from below. To see this, assume for simplicity that $N_0 = 0$, $N \ge 0$ and for some $n \in \mathbb{N}$ define $\sigma := \inf\{t \ge 0 \,|\, N_t \ge n\}$. Then on $\{t \ge \tau\}$ we have $N'_{t\wedge\sigma} = N_{t\wedge\sigma} - N_{\tau\wedge\sigma} \ge 0 - n$, since on $\{\tau < \sigma\}$ we have $N_{\tau\wedge\sigma} \le n$ and on $\{t \ge \tau \ge \sigma\}$ we have $N_{t\wedge\sigma} - N_{\tau\wedge\sigma} = N_\sigma - N_\sigma = 0$. $\diamondsuit$

Definition 5.4 For any payoff $X \in \mathbf{L}^\infty$, we call an $\mathcal{H}$-constrained portfolio (x, H, K) an $\mathcal{H}$*-constrained hedging portfolio* for X if its value process V is uniformly bounded from below and satisfies $V_T \geq X$. An $\mathcal{H}$-constrained portfolio $(\hat{x}, \hat{H}, \hat{K})$ for X with value process $\hat{V}$ is called *minimal $\mathcal{H}$-constrained hedging portfolio* for X if

$$\hat{V}_t \leq V_t \quad \text{for all } t \in [0, T]$$

for any $\mathcal{H}$-constrained hedging portfolio for X with value process V.

One central auxiliary result is a characterization of value processes corresponding to $\mathcal{H}$-constrained portfolios. For its formulation, we need to introduce some additional notation. Moreover, we make the following assumption to ensure that the market does not provide any arbitrage opportunities.

Assumption (NFLVR):
There exists $\hat{\mathbb{Q}} \in \mathcal{M}_1^e(\mathbb{P})$ such that S is a local $\hat{\mathbb{Q}}$-martingale.

Let us fix an admissible hedging set $\mathcal{H}$ and introduce the family of semimartingales

$$\mathcal{S} = \left\{ \int H \, dS \,\middle|\, H \in \mathcal{H} \right\}.$$

Definition 5.5 Let $\mathcal{P}(\mathcal{S})$ denote the class of all $\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P})$ for which there exists an increasing predictable process A (depending on $\mathbb{Q}$ and $\mathcal{S}$) such that $N - A$ is a local $\mathbb{Q}$-supermartingale for any $N \in \mathcal{S}$, i.e.,

$$A^N(\mathbb{Q}) \preceq A \text{ for all } N \in \mathcal{S}, \tag{5.1}$$

where $A^N(\mathbb{Q})$ is the predictable process of finite variation in the canonical decomposition of N under $\mathbb{Q}$. Then we call an increasing predictable process $A^{\mathcal{S}}(\mathbb{Q})$ the *upper variation process* of $\mathcal{S}$ under $\mathbb{Q}$ if it satisfies (5.1) and is minimal with respect to this property in the sense that $A^{\mathcal{S}}(\mathbb{Q}) \preceq A$ for any increasing predictable process A satisfying (5.1).

Remark 5.6 Note that (**NFLVR**) ensures that $\mathcal{P}(\mathcal{S}) \neq \emptyset$. In fact, since $\mathcal{H} \subseteq L^a_{\text{loc}}(S)$, if $\hat{\mathbb{Q}} \in \mathcal{M}_1^e(\mathbb{P})$ is a local martingale measure for S, then each $N \in \mathcal{S}$ is even a local $\hat{\mathbb{Q}}$-martingale by Corollary 3.5 in [AS94]. Hence $A^N(\hat{\mathbb{Q}}) \equiv 0$ so that $A^{\mathcal{S}}(\hat{\mathbb{Q}}) \equiv 0$. $\Diamond$

Example 5.7 If $\mathcal{H} = L^a_{\text{loc}}(S)$, i.e., in the case of unconstrained trading, it is well known that $\mathcal{P}(\mathcal{S})$ is just the set $\mathbb{M}^e$ of all equivalent local martingale measures for S. Indeed, this can also be seen from Remark 5.6 and Lemma 6.15 below. Moreover, as shown in Remark 5.6, we then have $A^{\mathcal{S}}(\mathbb{Q}) \equiv 0$ for all $\mathbb{Q} \in \mathcal{P}(\mathcal{S}) = \mathbb{M}^e$. Further examples can be found in [FK97]. $\Diamond$

Lemma 2.1 of [FK97], which characterizes $\mathcal{P}(\mathcal{S})$ and the upper variation processes $A^{\mathcal{S}}(\mathbb{Q})$, reads as follows:

Lemma 5.8 *A probability measure $\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P})$ belongs to $\mathcal{P}(\mathcal{S})$ if and only if all $N \in \mathcal{S}$ are a special semimartingales under $\mathbb{Q}$ and* $\operatorname*{ess\,sup}_{N \in \mathcal{S}} A^N(\mathbb{Q})_t < \infty$ *$\mathbb{P}$-a.s. for all*

$t \in [0,T]$. *In this case the upper variation process exists and is uniquely determined by the equations*

$$A^{\mathcal{S}}(\mathbb{Q})_\tau = \operatorname*{ess\,sup}_{N\in\mathcal{S}} A^N(\mathbb{Q})_\tau, \tag{5.2}$$

$$\mathbb{E}\left[A^{\mathcal{S}}(\mathbb{Q})_\tau\right] = \sup_{N\in\mathcal{S}} \mathbb{E}\left[A^N(\mathbb{Q})_\tau\right] \tag{5.3}$$

for all stopping times $\tau \le T$. Moreover, there exists a sequence $(N^n)_{n\in\mathbb{N}} \subseteq \mathcal{S}$ such that the compensators $A^n := A^{N^n}(\mathbb{Q})$ satisfy $A^n \preceq A^{n-1}$ and

$$\lim_{n\to\infty} \sup_{0\le t\le T} \left(A^{\mathcal{S}}(\mathbb{Q})_t - A^n_t\right) = 0 \quad \mathbb{P}-\text{a.s.}$$

Remark 5.9 Equation (5.3) is not really required for the characterization of $A^{\mathcal{S}}(\mathbb{Q})$ as it is a consequence of (5.2). In fact, Föllmer and Kramkov show in the proof of their Lemma 2.1 that for fixed $\mathbb{Q} \in \mathcal{P}(\mathcal{S})$, the space of compensators $\left\{ A^N(\mathbb{Q}) \,\middle|\, N \in \mathcal{S}\right\}$ is directed upwards. This implies in particular that $\left\{ A^N(\mathbb{Q})_\tau \,\middle|\, N \in \mathcal{S}\right\}$ is directed upwards for any stopping time $\tau \le T$ so that (5.2) implies (5.3). $\diamondsuit$

In order to manipulate the upper variation process we require the following result:

Lemma 5.10 *Fix a stopping time $\tau \le T$, a set $B \in \mathcal{F}_\tau$ and probability measures $\mathbb{Q}^1, \mathbb{Q}^2, \tilde{\mathbb{Q}} \in \mathcal{P}(\mathcal{S})$, and denote by $\tilde{Z}^1, \tilde{Z}^2$ the density processes of $\mathbb{Q}^1, \mathbb{Q}^2$ with respect to $\tilde{\mathbb{Q}}$. Then*

$$\frac{d\overline{\mathbb{Q}}}{d\tilde{\mathbb{Q}}} := \tilde{Z}^1_T \mathbf{1}_B + \tilde{Z}^1_\tau \frac{\tilde{Z}^2_T}{\tilde{Z}^2_\tau} \mathbf{1}_{B^c}$$

defines a probability measure $\overline{\mathbb{Q}} \in \mathcal{P}(\mathcal{S})$ such that $\overline{\mathbb{Q}} = \mathbb{Q}^1$ on $\mathcal{F}_\tau$ and

$$\mathbb{E}_{\overline{\mathbb{Q}}}[\,.\,|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}^1}[\,.\,|\mathcal{F}_t]\mathbf{1}_B + \mathbb{E}_{\mathbb{Q}^2}[\,.\,|\mathcal{F}_t]\mathbf{1}_{B^c} \quad \text{on } \{t > \tau\}. \tag{5.4}$$

The upper variation process of $\mathcal{S}$ under $\overline{\mathbb{Q}}$ can be written as

$$A^{\mathcal{S}}(\overline{\mathbb{Q}})_u = \left(\left(A^{\mathcal{S}}(\mathbb{Q}^1)_u - A^{\mathcal{S}}(\mathbb{Q}^1)_\tau\right)\mathbf{1}_B + \left(A^{\mathcal{S}}(\mathbb{Q}^2)_u - A^{\mathcal{S}}(\mathbb{Q}^2)_\tau\right)\mathbf{1}_{B^c}\right)\mathbf{1}_{\{u>\tau\}} + A^{\mathcal{S}}(\mathbb{Q}^1)_{u\wedge\tau}. \tag{5.5}$$

Proof That $\overline{\mathbb{Q}} = \mathbb{Q}^1$ on $\mathcal{F}_\tau$ is obvious. To see (5.4), denote by $\overline{Z}$ the density process of $\overline{\mathbb{Q}}$ with respect to $\tilde{\mathbb{Q}}$ and note that $\tilde{Z}^i_\tau \mathbf{1}_{\{t>\tau\}}$ is $\mathcal{F}_t$-measurable for $i = 1, 2$, so that

$$\overline{Z}_t \mathbf{1}_{\{t>\tau\}} = \left(\tilde{Z}^1_t \mathbf{1}_B + \tilde{Z}^1_\tau \frac{\tilde{Z}^2_t}{\tilde{Z}^2_\tau}\mathbf{1}_{B^c}\right)\mathbf{1}_{\{t>\tau\}}.$$

Then

$$\frac{\overline{Z}_T}{\overline{Z}_t}\mathbf{1}_{\{t>\tau\}} = \left(\frac{\tilde{Z}^1_T}{\tilde{Z}^1_t}\mathbf{1}_B + \frac{\tilde{Z}_T}{\tilde{Z}^2_t}\mathbf{1}_{B^c}\right)\mathbf{1}_{\{t>\tau\}}$$

yields (5.4). From this it is easy to check that for any $N \in \mathcal{S}$ the finite variation process in the canonical decomposition of N under $\overline{\mathbb{Q}}$ is given by

$$A^N(\overline{\mathbb{Q}})_u = \Big(\Big(A^N(\mathbb{Q}^1)_u - A^N(\mathbb{Q}^1)_\tau\Big)\,\mathbf{1}_B + \Big(A^N(\mathbb{Q}^2)_u - A^N(\mathbb{Q}^2)_\tau\Big)\,\mathbf{1}_{B^c}\Big)\,\mathbf{1}_{\{u>\tau\}} + A^N(\mathbb{Q}^1)_{u\wedge\tau}. \tag{5.6}$$

By Lemma 5.8, if $A^{\mathcal{S}}(\overline{\mathbb{Q}})$ exists, then it is given by (5.2). Hence (5.6) implies that $A^{\mathcal{S}}(\overline{\mathbb{Q}}) = A^{\mathcal{S}}(\mathbb{Q}^1)$ on the stochastic interval $[\![0,\tau]\!]$ and we are left to consider the increments after τ, i.e., to show that

$$\begin{aligned}&\Big(A^{\mathcal{S}}(\overline{\mathbb{Q}})_u - A^{\mathcal{S}}(\overline{\mathbb{Q}})_\tau\Big)\,\mathbf{1}_{\{u>\tau\}} \\ &= \Big(\Big(A^{\mathcal{S}}(\mathbb{Q}^1)_u - A^{\mathcal{S}}(\mathbb{Q}^1)_\tau\Big)\,\mathbf{1}_B + \Big(A^{\mathcal{S}}(\mathbb{Q}^2)_u - A^{\mathcal{S}}(\mathbb{Q}^2)_\tau\Big)\,\mathbf{1}_{B^c}\Big)\,\mathbf{1}_{\{u>\tau\}}.\end{aligned} \tag{5.7}$$

In preparation for this we note that for every $N^1, N^2 \in \mathcal{S}$, we also have $N^1\mathbf{1}_{[\![0,\tau]\!]} + N^2\mathbf{1}_{]\!]\tau,T]\!]} \in \mathcal{S}$ since $\mathcal{H}$ is predictably convex. This yields for any $\mathbb{Q} \in \mathcal{P}(\mathcal{S})$ that

$$\begin{aligned}\operatorname*{ess\,sup}_{N\in\mathcal{S}} A^N(\mathbb{Q})_u &= \operatorname*{ess\,sup}_{N\in\mathcal{S}} \Big\{A^N(\mathbb{Q})_u - A^N(\mathbb{Q})_\tau + A^N(\mathbb{Q})_\tau\Big\} \\ &= \operatorname*{ess\,sup}_{N\in\mathcal{S}} \Big\{A^N(\mathbb{Q})_u - A^N(\mathbb{Q})_\tau\Big\} + \operatorname*{ess\,sup}_{N\in\mathcal{S}} A^N(\mathbb{Q})_\tau\end{aligned}$$

on $\{u > \tau\}$ so that (5.2) and (5.6) imply that on $\{u > \tau\} \cap B^c$, we have

$$\begin{aligned}A^{\mathcal{S}}(\overline{\mathbb{Q}})_u &= \operatorname*{ess\,sup}_{N\in\mathcal{S}} \Big\{A^N(\mathbb{Q}^2)_u - A^N(\mathbb{Q}^2)_\tau + A^N(\mathbb{Q}^1)_\tau\Big\} \\ &= \operatorname*{ess\,sup}_{N\in\mathcal{S}} \Big\{A^N(\mathbb{Q}^2)_u - A^N(\mathbb{Q}^2)_\tau\Big\} + \operatorname*{ess\,sup}_{N\in\mathcal{S}} A^N(\mathbb{Q}^1)_\tau \\ &= \operatorname*{ess\,sup}_{N\in\mathcal{S}} \Big\{A^N(\mathbb{Q}^2)_u - A^N(\mathbb{Q}^2)_\tau\Big\} + A^{\mathcal{S}}(\mathbb{Q}^1)_\tau \\ &= A^{\mathcal{S}}(\mathbb{Q}^2)_u - A^{\mathcal{S}}(\mathbb{Q}^2)_\tau + A^{\mathcal{S}}(\mathbb{Q}^1)_\tau.\end{aligned}$$

As an analogous equality holds on $\{u > \tau\} \cap B$, this proves (5.7) and hence (5.5). In addition, existence of the upper variation process of $\mathcal{S}$ under $\overline{\mathbb{Q}}$ implies by Lemma 5.8 that $\overline{\mathbb{Q}} \in \mathcal{P}(\mathcal{S})$. □

One of our goals in the next section is the construction of a certain DMCUF from the minimal $\mathcal{H}$-constrained hedging portfolio. The key tool for this is the main result of this section, which is a slight modification of Proposition 4.1 in [FK97]:

Theorem 5.11 *For any $X \in \mathbf{L}^\infty$ there exists a minimal $\mathcal{H}$-constrained hedging portfolio $(\hat{x}, \hat{H}, \hat{K})$. Its value process equals*

$$\begin{aligned}\hat{V}_t &= \hat{x} + \int_0^t \hat{H}_s\,dS_s - \hat{K}_t \\ &= \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{P}(\mathcal{S})} \Big\{\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \mathbb{E}_{\mathbb{Q}}\Big[A^{\mathcal{S}}(\mathbb{Q})_T - A^{\mathcal{S}}(\mathbb{Q})_t\Big|\,\mathcal{F}_t\Big]\Big\}\end{aligned} \tag{5.8}$$

and is in particular uniformly bounded.

Remark 5.12 i) We can immediately see from Example 5.7 that in the unconstrained case, (5.8) becomes the well-known representation of the superhedging price process as

$$\hat{V}_t = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathbb{M}^e} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t].$$

ii) There are some differences between our work and [FK97]. First of all, Föllmer and Kramkov consider non-negative random variables as payoffs whereas we impose that payoffs are in $\mathbf{L}^\infty$. A more significant difference is that we allow the value process of $\mathcal{H}$-constrained hedging portfolios for a payoff X to be bounded from below by an arbitrary constant (depending on X), whereas Föllmer and Kramkov fix the lower bound at 0. This causes some changes in the results, and some arguments in the proof become a bit more involved.

$\diamondsuit$

The proof of Theorem 5.11 strongly relies on Theorem 4.1 of [FK97] which we state next:

Theorem 5.13 *Consider a process V which is locally bounded from below. Then the following statements are equivalent:*

a) *V is the value process of some $\mathcal{H}$-constrained portfolio (V_0, H, K), i.e.,*

$$V = V_0 + \int H\, dS - K.$$

b) *For all $\mathbb{Q} \in \mathcal{P}(\mathcal{S})$, the process $V - A^{\mathcal{S}}(\mathbb{Q})$ is a local $\mathbb{Q}$-supermartingale.*

As a second auxiliary result for the proof of Theorem 5.11, we require the following Lemma 5.14, which is similar to Lemma A.1 from [FK97].

Lemma 5.14 *For each $X \in \mathbf{L}^\infty$, there exists a uniformly bounded (RCLL adapted) process $V = (V_t)_{0\le t\le T}$ such that for all stopping times $\tau \le T$*

$$V_\tau = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{P}(\mathcal{S})} \left\{ \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_\tau] - \mathbb{E}_{\mathbb{Q}}\left[A^{\mathcal{S}}(\mathbb{Q})_T - A^{\mathcal{S}}(\mathbb{Q})_\tau \middle| \mathcal{F}_\tau \right] \right\} \quad \mathbb{P}\text{-a.s.} \tag{5.9}$$

Moreover, the process $V - A^{\mathcal{S}}(\tilde{\mathbb{Q}})$ is a local $\tilde{\mathbb{Q}}$-supermartingale for each $\tilde{\mathbb{Q}} \in \mathcal{P}(\mathcal{S})$.

Proof Define via the RHS of (5.9) a family of random variables U_τ, indexed by the set of all stopping times $\tau \le T$. Note that the family U_τ is uniformly bounded. Indeed, by (**NFLVR**) there exists an equivalent local martingale measure $\hat{\mathbb{Q}}$ for S so that in particular $\hat{\mathbb{Q}} \in \mathcal{P}(\mathcal{S})$ and $A^{\mathcal{S}}(\hat{\mathbb{Q}}) \equiv 0$. Then boundedness follows immediately from the definition of U_τ since

$$-\|X\|_{\mathbf{L}^\infty} \le \mathbb{E}_{\hat{\mathbb{Q}}}[X|\mathcal{F}_\tau] \le U_\tau \le \|X\|_{\mathbf{L}^\infty}. \tag{5.10}$$

.

1) Fix $\tilde{\mathbb{Q}} \in \mathcal{P}(\mathcal{S})$, $t \in [0,T]$ and stopping times $\sigma \le t$, $\tau \le T$ such that the stopped upper variation process $A^{\mathcal{S}}(\tilde{\mathbb{Q}})^\tau$ is bounded. We first show that

$$\begin{aligned}&\mathbb{E}_{\tilde{\mathbb{Q}}}[U_{t\wedge\tau}|\mathcal{F}_\sigma]\\ &\quad= \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{P}(\mathcal{S})_{t\wedge\tau}} \mathbb{E}_{\mathbb{Q}}\left[X - A^{\mathcal{S}}(\mathbb{Q})_T\middle|\mathcal{F}_{\sigma\wedge\tau}\right] + \mathbb{E}_{\tilde{\mathbb{Q}}}\left[A^{\mathcal{S}}(\tilde{\mathbb{Q}})_{t\wedge\tau}\middle|\mathcal{F}_{\sigma\wedge\tau}\right],\end{aligned} \tag{5.11}$$

where

$$\mathcal{P}(\mathcal{S})_{t\wedge\tau} := \left\{\mathbb{Q}\in\mathcal{P}(\mathcal{S}) \,\middle|\, \mathbb{Q} = \tilde{\mathbb{Q}} \text{ on } \mathcal{F}_{t\wedge\tau}\right\}.$$

For abbreviation we introduce on $\mathcal{P}(\mathcal{S})$ the operator

$$F(\mathbb{Q}) := \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_{t\wedge\tau}] - \mathbb{E}_{\mathbb{Q}}\left[A^{\mathcal{S}}(\mathbb{Q})_T - A^{\mathcal{S}}(\mathbb{Q})_{t\wedge\tau}\middle|\mathcal{F}_{t\wedge\tau}\right]. \tag{5.12}$$

Moreover, in part 1) of this proof we express all densities and density processes with respect to $\tilde{\mathbb{Q}}$. To see (5.11), we first note that

$$U_{t\wedge\tau} = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{P}(\mathcal{S})} F(\mathbb{Q}) = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{P}(\mathcal{S})_{t\wedge\tau}} F(\mathbb{Q}). \tag{5.13}$$

In fact, take $\mathbb{Q}\in\mathcal{P}(\mathcal{S})$ with density process $(Z_u)_{0\le u\le T}$ and define a new measure $\overline{\mathbb{Q}}$ by the density

$$\overline{Z}_T := \frac{Z_T}{Z_{t\wedge\tau}} \tag{5.14}$$

with respect to $\tilde{\mathbb{Q}}$. Then Lemma 5.10 implies that $\overline{\mathbb{Q}} \in \mathcal{P}(\mathcal{S})_{t\wedge\tau}$ and

$$A^{\mathcal{S}}(\overline{\mathbb{Q}})_u = \left(A^{\mathcal{S}}(\mathbb{Q})_u - A^{\mathcal{S}}(\mathbb{Q})_{t\wedge\tau}\right)\mathbf{1}_{\{u>(t\wedge\tau)\}} + A^{\mathcal{S}}(\tilde{\mathbb{Q}})_{\{u\wedge(t\wedge\tau)\}},$$

and we have $\mathbb{E}_{\overline{\mathbb{Q}}}[\,.\,|\mathcal{F}_{t\wedge\tau}] = \mathbb{E}_{\mathbb{Q}}[\,.\,|\mathcal{F}_{t\wedge\tau}]$ so that (5.13) holds. Next we note that the set $\{\,F(\mathbb{Q}) \mid \mathbb{Q}\in\mathcal{P}(\mathcal{S})_{t\wedge\tau}\}$ is a lattice, since for any $B\in\mathcal{F}_{t\wedge\tau}$ and $\mathbb{Q}^1,\mathbb{Q}^2 \in \mathcal{P}(\mathcal{S})_{t\wedge\tau}$ with densities Z^1_T, Z^2_T we can define a probability measure $\mathbb{Q}'$ by $Z'_T := Z^1_T\mathbf{1}_B + Z^2_T\mathbf{1}_{B^c}$ to obtain from Lemma 5.10 that $\mathbb{Q}'\in\mathcal{P}(\mathcal{S})_{t\wedge\tau}$ and $F(\mathbb{Q}') = F(\mathbb{Q}^1)\mathbf{1}_B + F(\mathbb{Q}^2)\mathbf{1}_{B^c}$. This guarantees ([Nev75]) by (5.13) the existence of some sequence $(\mathbb{Q}^m)_{m\in\mathbb{N}} \subseteq \mathcal{P}(\mathcal{S})_{t\wedge\tau}$ such that

$$U_{t\wedge\tau} = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{P}(\mathcal{S})_{t\wedge\tau}} F(\mathbb{Q}) = \nearrow - \lim_{m\to\infty} F(\mathbb{Q}^m). \tag{5.15}$$

To finish the proof of (5.11), we recall that by (**NFLVR**) there exists an equivalent local martingale measure $\hat{\mathbb{Q}}$ for S. By Remark 5.6, we have $\hat{\mathbb{Q}}\in\mathcal{P}(\mathcal{S})$ and $A^{\mathcal{S}}(\hat{\mathbb{Q}}) \equiv 0$. If we denote the density process of $\hat{\mathbb{Q}}$ by $\hat{Z}$, we can define as in (5.14) a probability measure $\overline{\mathbb{Q}}\in\mathcal{P}(\mathcal{S})_{t\wedge\tau}$ by the density

$$\overline{Z}_T := \frac{\hat{Z}_T}{\hat{Z}_{t\wedge\tau}}.$$

Since $\{\,F(\mathbb{Q}) \mid \mathbb{Q}\in\mathcal{P}(\mathcal{S})_{t\wedge\tau}\}$ is a lattice, we can assume without loss of generality that in the above sequence $\mathbb{Q}^1 = \overline{\mathbb{Q}}$. By Lemma 5.10, $A^{\mathcal{S}}(\overline{\mathbb{Q}})_T - A^{\mathcal{S}}(\overline{\mathbb{Q}})_{t\wedge\tau} =$

$A^{\mathcal{S}}(\hat{\mathbb{Q}})_T - A^{\mathcal{S}}(\hat{\mathbb{Q}})_{t\wedge\tau} = 0$ so that we can apply the monotone convergence theorem to obtain from (5.15), (5.12) and since $\tilde{\mathbb{Q}} = \mathbb{Q}^m$ on $\mathcal{F}_{t\wedge\tau}$ that

$$\begin{aligned} \mathbb{E}_{\tilde{\mathbb{Q}}}[U_{t\wedge\tau}|\mathcal{F}_\sigma] &= \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\nearrow - \lim_{m\to\infty} F(\mathbb{Q}^m)\middle|\mathcal{F}_\sigma\right] \\ &= \nearrow - \lim_{m\to\infty} \mathbb{E}_{\tilde{\mathbb{Q}}}\left[F(\mathbb{Q}^m)|\,\mathcal{F}_\sigma\right] \\ &= \nearrow - \lim_{m\to\infty} \mathbb{E}_{\mathbb{Q}^m}\left[F(\mathbb{Q}^m)|\,\mathcal{F}_\sigma\right] \\ &\leq \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{P}(\mathcal{S})_{t\wedge\tau}} \mathbb{E}_{\mathbb{Q}}\left[X - A^{\mathcal{S}}(\mathbb{Q})_T + A^{\mathcal{S}}(\mathbb{Q})_{t\wedge\tau}\middle|\mathcal{F}_{\sigma\wedge\tau}\right]. \end{aligned} \tag{5.16}$$

As the converse inequality is trivial due to (5.13), we even get equality in (5.16). This implies (5.11) since for any $\mathbb{Q} \in \mathcal{P}_{t\wedge\tau}$ with density process Z we have $\mathbb{Q} = \tilde{\mathbb{Q}}$ on $\mathcal{F}_{t\wedge\tau}$ and $Z_T = \frac{Z_T}{Z_{t\wedge\tau}}$ so that by Lemma 5.10

$$A^{\mathcal{S}}(\mathbb{Q})_{t\wedge\tau} = A^{\mathcal{S}}(\tilde{\mathbb{Q}})_{t\wedge\tau}. \tag{5.17}$$

2) As in 1), we fix $\tilde{\mathbb{Q}} \in \mathcal{P}(\mathcal{S})$, $t \in [0,T]$ and stopping times $\sigma \leq t$, $\tau \leq T$ such that the stopped process $A^{\mathcal{S}}(\tilde{\mathbb{Q}})^\tau$ is uniformly bounded. We show the following supermartingale property for the family $\left(U_{t\wedge\tau} - A^{\mathcal{S}}(\tilde{\mathbb{Q}})_{t\wedge\tau}\right)_{0\leq t\leq T}$:

$$\mathbb{E}_{\tilde{\mathbb{Q}}}\left[U_{t\wedge\tau} - A^{\mathcal{S}}(\tilde{\mathbb{Q}})_{t\wedge\tau}\middle|\mathcal{F}_\sigma\right] \leq U_{\sigma\wedge\tau} - A^{\mathcal{S}}(\tilde{\mathbb{Q}})_{\sigma\wedge\tau}. \tag{5.18}$$

Indeed, since $\sigma \leq t$ implies that $\mathcal{P}(\mathcal{S})_{t\wedge\tau} \subseteq \mathcal{P}(\mathcal{S})_{\sigma\wedge\tau}$, we get from (5.11) that

$$\begin{aligned} &\mathbb{E}_{\tilde{\mathbb{Q}}}[U_{t\wedge\tau}|\mathcal{F}_\sigma] \\ &\quad\leq \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{P}(\mathcal{S})_{\sigma\wedge\tau}} \mathbb{E}_{\mathbb{Q}}\left[X - A^{\mathcal{S}}(\mathbb{Q})_T\middle|\mathcal{F}_{\sigma\wedge\tau}\right] + \mathbb{E}_{\tilde{\mathbb{Q}}}\left[A^{\mathcal{S}}(\tilde{\mathbb{Q}})_{t\wedge\tau}\middle|\mathcal{F}_{\sigma\wedge\tau}\right]. \end{aligned} \tag{5.19}$$

Because $A^{\mathcal{S}}(\tilde{\mathbb{Q}})_{t\wedge\tau}$ is $\mathcal{F}_\tau$-measurable, we have

$$\begin{aligned} \mathbb{E}_{\tilde{\mathbb{Q}}}\left[A^{\mathcal{S}}(\tilde{\mathbb{Q}})_{t\wedge\tau}\middle|\mathcal{F}_{\sigma\wedge\tau}\right] &= \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\mathbb{E}_{\tilde{\mathbb{Q}}}\left[A^{\mathcal{S}}(\tilde{\mathbb{Q}})_{t\wedge\tau}\middle|\mathcal{F}_\tau\right]\middle|\mathcal{F}_\sigma\right] \\ &= \mathbb{E}_{\tilde{\mathbb{Q}}}\left[A^{\mathcal{S}}(\tilde{\mathbb{Q}})_{t\wedge\tau}\middle|\mathcal{F}_\sigma\right]. \end{aligned}$$

From this together with (5.19), (5.13) and (5.17), we get

$$\begin{aligned} \mathbb{E}_{\tilde{\mathbb{Q}}}\left[U_{t\wedge\tau} - A^{\mathcal{S}}(\tilde{\mathbb{Q}})_{t\wedge\tau}\middle|\mathcal{F}_\sigma\right] &\leq \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{P}(\mathcal{S})_{\sigma\wedge\tau}} \mathbb{E}_{\mathbb{Q}}\left[X - A^{\mathcal{S}}(\mathbb{Q})_T\middle|\mathcal{F}_{\sigma\wedge\tau}\right] \\ &= U_{\sigma\wedge\tau} - A^{\mathcal{S}}(\tilde{\mathbb{Q}})_{\sigma\wedge\tau} \end{aligned} \tag{5.20}$$

and hence (5.18).

3) Our next goal is to show that for a sequence of stopping times $\sigma_n \leq T$ decreasing to another stopping time $\sigma \leq T$, we have

$$\mathbb{E}_{\hat{\mathbb{Q}}}[U_\sigma] = \lim_{n\to\infty} \mathbb{E}_{\hat{\mathbb{Q}}}[U_{\sigma_n}] \tag{5.21}$$

for any equivalent local martingale measure $\hat{\mathbb{Q}}$ for S. Indeed, (5.18) yields for $\tilde{\mathbb{Q}} = \hat{\mathbb{Q}}$, $t = T$ and $\tau = \sigma_n$ that

$$U_\sigma \geq \mathbb{E}_{\hat{\mathbb{Q}}}[U_{\sigma_n}|\mathcal{F}_\sigma] \tag{5.22}$$

and hence also

$$\mathbb{E}_{\hat{\mathbb{Q}}}[U_\sigma] \geq \limsup_{n\to\infty} \mathbb{E}_{\hat{\mathbb{Q}}}[U_{\sigma_n}]. \tag{5.23}$$

To prove the converse inequality, we fix $\epsilon > 0$. From (5.11), we get for $\tilde{\mathbb{Q}} = \hat{\mathbb{Q}}$, $\tau = \sigma$, $t = T$ and $\sigma = 0$ there that

$$\mathbb{E}_{\hat{\mathbb{Q}}}[U_\sigma] = \sup_{\mathbb{Q}\in\hat{\mathcal{P}}(\mathcal{S})_\sigma} \mathbb{E}_{\mathbb{Q}}\left[X - A^{\mathcal{S}}(\mathbb{Q})_T\right],$$

where $\hat{\mathcal{P}}(\mathcal{S})_\sigma = \{\mathbb{Q} \in \mathcal{P}(\mathcal{S}) \,|\, \mathbb{Q} = \hat{\mathbb{Q}} \text{ on } \mathcal{F}_\sigma\}$. Hence there exists some $\mathbb{Q}' \in \hat{\mathcal{P}}(\mathcal{S})_\sigma$ such that

$$\mathbb{E}_{\hat{\mathbb{Q}}}[U_\sigma] \leq \mathbb{E}_{\mathbb{Q}'}[X - A^{\mathcal{S}}(\mathbb{Q}')_T] + \epsilon. \tag{5.24}$$

Note that in particular $0 \leq A^{\mathcal{S}}(\mathbb{Q}')_T \in \mathbf{L}^1(\mathbb{Q}')$. For the rest of this proof, all densities and density processes are now expressed with respect to $\hat{\mathbb{Q}}$. Denote by $Z' = (Z'_s)_{0\leq s\leq T}$ the density process of $\mathbb{Q}'$ and define $\nu := \inf\{u > 0 \mid Z'_u \leq 0.1\} \wedge T$. As Z' is right-continuous and $Z' = 1$ on $[\![0, \sigma]\!]$, we have $\nu > \sigma$. For each $n \in I\!N$ we define a measure $\mathbb{Q}^n$ by

$$Z^n_T := \frac{Z'_T}{Z'_{\sigma_n}} \mathbf{1}_{\{\sigma_n<\nu\}} + \mathbf{1}_{\{\sigma_n\geq\nu\}}.$$

By Lemma 5.10, $\mathbb{Q}^n \in \hat{\mathcal{P}}(\mathcal{S})_{\sigma_n}$ and

$$A^{\mathcal{S}}(\mathbb{Q}^n)_T = \left(A^{\mathcal{S}}(\mathbb{Q}')_T - A^{\mathcal{S}}(\mathbb{Q}')_{\sigma_n}\right) \mathbf{1}_{\{\sigma_n<\nu\}}.$$

Hence we can apply (5.11) for each $n \in I\!N$ with $\tilde{\mathbb{Q}} = \hat{\mathbb{Q}}$, $t = T$, $\sigma = 0$ and $\tau = \sigma_n$ to obtain

$$\begin{aligned} \liminf_{n\to\infty} \mathbb{E}_{\hat{\mathbb{Q}}}[U_{\sigma_n}] &\geq \liminf_{n\to\infty} \mathbb{E}_{\mathbb{Q}^n}\left[X - A^{\mathcal{S}}(\mathbb{Q}^n)_T\right] \\ &\geq \liminf_{n\to\infty} \mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{Z'_T}{Z'_{\sigma_n}}\left(X - A^{\mathcal{S}}(\mathbb{Q}')_T + A^{\mathcal{S}}(\mathbb{Q}')_{\sigma_n}\right)\mathbf{1}_{\{\sigma_n<\nu\}}\right] \\ &\quad + \liminf_{n\to\infty} \mathbb{E}_{\hat{\mathbb{Q}}}[X\mathbf{1}_{\{\sigma_n\geq\nu\}}]. \end{aligned} \tag{5.25}$$

Because $\sigma_n \searrow \sigma < \nu$, the second summand is zero by dominated convergence. For the first summand, we note that $Z'_{\sigma_n} > 0.1$ on $\{\sigma_n < \nu\}$, so that a lower bound for the sequence $\left(\frac{1}{Z_{\sigma_n}}\left(X - A^{\mathcal{S}}(\mathbb{Q}')_T + A^{\mathcal{S}}(\mathbb{Q}')_{\sigma_n}\right)\mathbf{1}_{\{\sigma_n<\nu\}}\right)_{n\in I\!N}$ is given by $10\left(-\|X\|_{\mathbf{L}^\infty} - A^{\mathcal{S}}(\mathbb{Q}')_T\right) \in \mathbf{L}^1(\mathbb{Q}')$. This allows us to apply Fatou's lemma to get from (5.25), (5.24) and since $\hat{\mathbb{Q}} = \mathbb{Q}'$ on $\mathcal{F}_\sigma$ that

$$\begin{aligned} \liminf_{n\to\infty} \mathbb{E}_{\hat{\mathbb{Q}}}[U_{\sigma_n}] &\geq \mathbb{E}_{\mathbb{Q}'}\left[X - A^{\mathcal{S}}(\mathbb{Q}')_T + A^{\mathcal{S}}(\mathbb{Q}')_\sigma\right] \\ &= \mathbb{E}_{\mathbb{Q}'}\left[X - A^{\mathcal{S}}(\mathbb{Q}')_T\right] \\ &\geq \mathbb{E}_{\hat{\mathbb{Q}}}[U_\sigma] - \epsilon, \end{aligned} \tag{5.26}$$

where we used analogously to (5.17) that $\mathbb{Q}' \in \mathcal{P}(\mathcal{S})_\sigma$ so that $A^{\mathcal{S}}(\mathbb{Q}')_\sigma = A^{\mathcal{S}}(\hat{\mathbb{Q}})_\sigma = 0$. Since $\epsilon > 0$ was arbitrary, this together with (5.23) implies (5.21).

4) Next we deduce that $U := (U_t)_{0 \le t \le T}$ admits an RCLL modification V. Denote by $\hat{\mathbb{Q}}$ an equivalent local martingale measure for S and observe that with $s \le t \le T$, (5.18) yields for $\tilde{\mathbb{Q}} = \hat{\mathbb{Q}}$, $\sigma = s$ and $\tau = T$ that

$$\mathbb{E}_{\hat{\mathbb{Q}}}[U_t | \mathcal{F}_s] \le U_s.$$

Hence the family $U = (U_t)_{0 \le t \le T}$ satisfies under $\hat{\mathbb{Q}}$ the supermartingale property and is by (5.21) right-continuous in expectation. This implies by Theorem VI.3 of [DM82] the existence of an RCLL modification $V = (V_t)_{0 \le t \le T}$ of U.

5) By the definition of a right-continuous modification, (5.9) holds for any deterministic time t. However, we still have to show that it remains true for any stopping time $\sigma \le T$. To see this, take a sequence of stopping times $\sigma_n \le T$ decreasing to σ and taking only rational values. If we can show that

$$\lim_{n \to \infty} \mathbb{E}_{\hat{\mathbb{Q}}}[\,|U_\sigma - U_{\sigma_n}|\,] = 0, \tag{5.27}$$

then we are done since right-continuity and boundedness of V then imply that

$$U_\sigma = \lim_{n \to \infty} U_{\sigma_n} = \lim_{n \to \infty} V_{\sigma_n} = V_\sigma,$$

where the limits are taken in $\mathbf{L}^1(\hat{\mathbb{Q}})$. However, (5.18) yields for $\tilde{\mathbb{Q}} = \hat{\mathbb{Q}}$, $t = T$, $\sigma = \sigma_{n+1}$ there and $\tau = \sigma_n$ that

$$\mathbb{E}_{\hat{\mathbb{Q}}}[U_{\sigma_n} | \mathcal{F}_{\sigma_{n+1}}] \le U_{\sigma_{n+1}}.$$

Since $(\sigma_n)_{n \in I\!N}$ is decreasing and U is uniformly bounded, this means that $(U_{\sigma_n})_{n \in I\!N}$ is a backward supermartingale under $\hat{\mathbb{Q}}$. By Theorem V.30 of [DM82], $(U_{\sigma_n})_{n \in I\!N}$ therefore converges in $\mathbf{L}^1(\hat{\mathbb{Q}})$ to some $\overline{U}$. Clearly $\overline{U}$ is measurable with respect to $\mathcal{F}_\sigma = \bigcap_{n \in I\!N} \mathcal{F}_{\sigma_n}$ so that the sequence $(\mathbb{E}_{\hat{\mathbb{Q}}}[U_{\sigma_n} | \mathcal{F}_\sigma])_{n \in I\!N}$ also converges to $\overline{U}$ in $\mathbf{L}^1(\hat{\mathbb{Q}})$, and so it remains to show that $(\mathbb{E}_{\hat{\mathbb{Q}}}[U_{\sigma_n} | \mathcal{F}_\sigma])_{n \in I\!N}$ converges to U_σ in $\mathbf{L}^1(\hat{\mathbb{Q}})$. But this follows immediately from (5.21) and (5.22).

6) Finally we want to conclude that $V - A^{\mathcal{S}}(\tilde{\mathbb{Q}})$ is a local $\tilde{\mathbb{Q}}$-supermartingale for each $\tilde{\mathbb{Q}} \in \mathcal{P}(\mathcal{S})$. To that end let $(\tau_n)_{n \in I\!N}$ be a localizing sequence such that the upper variation process $A^{\mathcal{S}}(\tilde{\mathbb{Q}})^{\tau_n}$ is bounded for each n. Then 5) implies that

$$V_t^{\tau_n} - A^{\mathcal{S}}(\tilde{\mathbb{Q}})_t^{\tau_n} = U_{t \wedge \tau_n} - A^{\mathcal{S}}(\tilde{\mathbb{Q}})_{t \wedge \tau_n},$$

which together with 2) and boundedness of V implies that $(V - A^{\mathcal{S}}(\tilde{\mathbb{Q}}))^{\tau_n}$ is a bounded $\tilde{\mathbb{Q}}$-supermartingale.

□

Proof of Theorem 5.11 Use Lemma 5.14 to define $\hat{V}$ as a uniformly bounded (RCLL) process satisfying for each $t \in [0, T]$

$$\hat{V}_t = \operatorname*{ess\,sup}_{\mathbb{Q} \in \mathcal{P}(\mathcal{S})} \left\{ \mathbb{E}_{\mathbb{Q}} \left[X - A^{\mathcal{S}}(\mathbb{Q})_T + A^{\mathcal{S}}(\mathbb{Q})_t \middle| \mathcal{F}_t \right] \right\}. \tag{5.28}$$

Then Lemma 5.14 and Theorem 5.13 imply that $\hat{V}$ is the value process of some $\mathcal{H}$-constrained hedging portfolio $(\hat{x}, \hat{H}, \hat{K})$ for X. To prove that $\hat{V}$ is minimal, we first show that in (5.28) we can replace $\mathcal{P}(\mathcal{S})$ by

$$\mathcal{P}(\mathcal{S})^b := \left\{ \mathbb{Q} \in \mathcal{P}(\mathcal{S}) \;\middle|\; \mathbb{E}_{\mathbb{Q}} \left[-A^{\mathcal{S}}(\mathbb{Q})_T + A^{\mathcal{S}}(\mathbb{Q})_t \,\middle|\, \mathcal{F}_t \right] \geq -2\|X\|_{\mathbf{L}^\infty} - 1 \right\}.$$

By (**NFLVR**) there exists an equivalent local martingale measure $\hat{\mathbb{Q}}$ for S. As $A^{\mathcal{S}}(\hat{\mathbb{Q}}) \equiv 0$, we have $\hat{\mathbb{Q}} \in \mathcal{P}(\mathcal{S})^b$. Since $\hat{V}_t \geq \mathbb{E}_{\hat{\mathbb{Q}}}[X|\mathcal{F}_t] \geq -\|X\|_{\mathbf{L}^\infty}$ and $X \leq \|X\|_{\mathbf{L}^\infty}$, we claim that a measure $\mathbb{Q} \in \mathcal{P}(\mathcal{S})$ cannot contribute to the essential supremum in (5.28) on the set

$$B := \left\{ \mathbb{E}_{\mathbb{Q}} \left[-A^{\mathcal{S}}(\mathbb{Q})_T + A^{\mathcal{S}}(\mathbb{Q})_t \,\middle|\, \mathcal{F}_t \right] < -2\|X\|_{\mathbf{L}^\infty} - 1 \right\} \in \mathcal{F}_t.$$

In fact, if $Z = (Z_t)_{0 \leq t \leq T}$ denotes the density process of $\mathbb{Q}$ with respect to $\hat{\mathbb{Q}}$, we can construct a measure $\overline{\mathbb{Q}}$ via its density

$$\overline{Z}_T := \mathbf{1}_B + \frac{Z_T}{Z_t} \mathbf{1}_{B^c}$$

with respect to $\hat{\mathbb{Q}}$ to obtain from Lemma 5.10 that $\overline{\mathbb{Q}} \in \mathcal{P}(\mathcal{S})$ and that

$$\begin{aligned}
&\mathbb{E}_{\overline{\mathbb{Q}}} \left[-A^{\mathcal{S}}(\overline{\mathbb{Q}})_T + A^{\mathcal{S}}(\overline{\mathbb{Q}})_t \,\middle|\, \mathcal{F}_t \right] \\
&\quad = \mathbb{E}_{\hat{\mathbb{Q}}} \left[-A^{\mathcal{S}}(\hat{\mathbb{Q}})_T + A^{\mathcal{S}}(\hat{\mathbb{Q}})_t \,\middle|\, \mathcal{F}_t \right] \mathbf{1}_B + \mathbb{E}_{\mathbb{Q}} \left[-A^{\mathcal{S}}(\mathbb{Q})_T + A^{\mathcal{S}}(\mathbb{Q})_t \,\middle|\, \mathcal{F}_t \right] \mathbf{1}_{B^c} \\
&\quad \geq -2\|X\|_{\mathbf{L}^\infty} - 1,
\end{aligned}$$

where the inequality holds by the definition of B and because $A^{\mathcal{S}}(\hat{\mathbb{Q}}) \equiv 0$. This shows that $\overline{\mathbb{Q}}$ is in $\mathcal{P}(\mathcal{S})^b$ and also that in (5.28) we can indeed replace $\mathcal{P}(\mathcal{S})$ by $\mathcal{P}(\mathcal{S})^b$ since

$$\begin{aligned}
&\mathbb{E}_{\hat{\mathbb{Q}}} \left[-A^{\mathcal{S}}(\hat{\mathbb{Q}})_T + A^{\mathcal{S}}(\hat{\mathbb{Q}})_t \,\middle|\, \mathcal{F}_t \right] \mathbf{1}_B + \mathbb{E}_{\mathbb{Q}} \left[-A^{\mathcal{S}}(\mathbb{Q})_T + A^{\mathcal{S}}(\mathbb{Q})_t \,\middle|\, \mathcal{F}_t \right] \mathbf{1}_{B^c} \\
&\quad \geq \mathbb{E}_{\mathbb{Q}} \left[-A^{\mathcal{S}}(\mathbb{Q})_T + A^{\mathcal{S}}(\mathbb{Q})_t \,\middle|\, \mathcal{F}_t \right],
\end{aligned}$$

where we used again that $A^{\mathcal{S}}(\hat{\mathbb{Q}}) \equiv 0$.

Now we can prove that $\hat{V}$ is a lower bound for the value process V of any $\mathcal{H}$-constrained hedging portfolio (x, H, K) for X. To that end fix $\mathbb{Q} \in \mathcal{P}(\mathcal{S})^b$ and let $(\tau_n)_{n \in I\!N}$ be a localizing sequence of stopping times such that $A^{\mathcal{S}}(\mathbb{Q})$ is bounded on $[\![0, \tau_n]\!]$. By the definition of $A^{\mathcal{S}}(\mathbb{Q})$, the process $V - A^{\mathcal{S}}(\mathbb{Q})$ is a local $\mathbb{Q}$-supermartingale. On $[\![0, \tau_n]\!]$, it is bounded from below and hence a $\mathbb{Q}$-supermartingale, and therefore

$$V_{t \wedge \tau_n} \geq \mathbb{E}_{\mathbb{Q}} \left[V_{\tau_n} - A^{\mathcal{S}}(\mathbb{Q})_{\tau_n} + A^{\mathcal{S}}(\mathbb{Q})_{t \wedge \tau_n} \,\middle|\, \mathcal{F}_t \right]$$

for each $n \in I\!N$. Moreover, $\mathbb{Q} \in \mathcal{P}(\mathcal{S})^b$ implies that $-A^{\mathcal{S}}(\mathbb{Q})_T + A^{\mathcal{S}}(\mathbb{Q})_t$ is $\mathbb{Q}$-integrable and hence an integrable lower bound for $\left(-A^{\mathcal{S}}(\mathbb{Q})_{\tau_n} + A^{\mathcal{S}}(\mathbb{Q})_{t \wedge \tau_n}\right)_{n \in I\!N}$. This allows us to apply Fatou's lemma to obtain

$$\begin{aligned}
V_t &\geq \mathbb{E}_{\mathbb{Q}} \left[\liminf_{n \to \infty} \left(V_{\tau_n} - A^{\mathcal{S}}(\mathbb{Q})_{\tau_n} + A^{\mathcal{S}}(\mathbb{Q})_{t \wedge \tau_n} \right) \middle|\, \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}} \left[V_T - A^{\mathcal{S}}(\mathbb{Q})_T + A^{\mathcal{S}}(\mathbb{Q})_t \,\middle|\, \mathcal{F}_t \right] \\
&\geq \mathbb{E}_{\mathbb{Q}} \left[X - A^{\mathcal{S}}(\mathbb{Q})_T + A^{\mathcal{S}}(\mathbb{Q})_t \,\middle|\, \mathcal{F}_t \right].
\end{aligned}$$

Because $\mathbb{Q} \in \mathcal{P}(\mathcal{S})^b$ was arbitrary, this implies

$$V_t \geq \operatorname*{ess\,sup}_{\mathbb{Q} \in \mathcal{P}(\mathcal{S})^b} \mathbb{E}_{\mathbb{Q}} \left[X - A^{\mathcal{S}}(\mathbb{Q})_T + A^{\mathcal{S}}(\mathbb{Q})_t \middle| \mathcal{F}_t \right] = \hat{V}_t.$$

□

6 Dynamic Utility Indifference Valuation

Asset valuation in incomplete markets is still an important problem in mathematical finance. One approach is the dynamic utility indifference valuation method which we consider in this section. After defining the utility indifference value for each time $t \in [0,T]$, we investigate its properties as a functional on $\mathbf{L}^\infty$, in particular with respect to continuity and time-consistency. For this we observe that the indifference valuation functional is obtained by normalization of the convolution of the DMCUF corresponding to the agent's preferences and the *market DMCUF* whose acceptance sets consist (up to sign) of exactly those payoffs that can be superhedged at zero cost. We extend an idea of Föllmer/Schied [FS02] by using the optional decomposition under constraints dynamically over time to construct the market DMCUF, and notably show that this DMCUF is strongly time-consistent. Moreover, we discuss the connections between this utility indifference valuation approach and arbitrage opportunities, explain the link to good-deal bounds, and examine the special case when trading in the market is possible without constraints.

Valuation by indifference with respect to an expected utility is an old theme and has been much studied again in the last years. An early reference is Hodges/Neuberger [HN89]; Frittelli [Fri00] and Rouge/El Karoui [REK00] are at the start of the recent resurgence of activity, and Becherer [Bec03] and Henderson/Hobson [HH04] contain overviews and many more references. However, explicit results are hard to obtain because except for the exponential case, the utility-based certainty equivalent is not translation invariant.

The idea of replacing expected utility by a monetary (hence translation invariant) utility functional and the naturally ensuing link to the convolution with the market functional have only emerged rather recently. Perhaps the earliest reference which is based on a similar idea can be found in a general abstract (but static) form is Jaschke/Küchler [JK01], even though the formulation there is for coherent risk measures and cast in terms of good-deal bounds. Indifference valuation proper is briefly mentioned in [BEK05] and discussed in more detail in Xu [Xu05] which also contains a number of worked examples. However, both deal only with the static case, and [Xu05] has no constraints in the market. Larsen/Pirvu/Shreve/Tütüncü [LPST05] contains a dynamic treatment for a particular class of examples where Φ is given via a finite set of scenario and stress measures, generalizing an idea from Carr/Geman/Madan [CGM01]. None of these works study the issue of time-consistency.

The underlying idea is the following. For each $t \in [0,T]$, let U_t be a functional which maps $\mathbf{L}^\infty$ into $\mathbf{L}^\infty(\mathcal{F}_t)$. We assume that $U_t(X)$ models the utility that some (fixed) agent assigns at time $t \leq T$ to the payoff X which is due at time T. We suppose that she can trade in a financial market and denote by $\mathcal{C}_t$ the set of payoffs due at time T that she can superhedge by trading during $(t,T]$ with zero initial capital. If the agent has at time

t an initial endowment $x_t \in \mathbf{L}^\infty(\mathcal{F}_t)$, she can implicitly determine a time t value $p_t(X)$ for the payoff $X \in \mathbf{L}^\infty$ by the indifference requirement

$$\operatorname*{ess\,sup}_{g\in\mathcal{C}_t} U_t(x_t + g) = \operatorname*{ess\,sup}_{g\in\mathcal{C}_t} U_t(x_t - p_t(X) + g + X) \tag{6.1}$$

(presuming that $p_t(X)$ is well-defined). We call $p(X) = (p_t(X))_{0\le t\le T}$ the *utility indifference value process* for X since it makes the agent at each time t indifferent (according to U_t) between buying the asset X or not, provided that she always optimally exploits her trading opportunities.

Remark 6.1 i) The set $\mathcal{C}_t$ consists of all payoffs that the agent can superhedge by trading during $(t, T]$ from zero initial capital. Hence $\mathcal{C}_t$ is solid. This will be required later when we assume that $-\mathcal{C}_t$ is an acceptable set. Note that we assumed implicitly in the definition of p that the initial endowment x_t can be transferred from t to T, i.e., the existence of a bank account with zero interest rate. However, besides from this, we did not impose any conditions on the structure of $\mathcal{C}_t$ so far. In fact, $\mathcal{C}_t$ can be used to incorporate transaction costs or bid and ask prices for the traded assets. However, when we specify $\mathcal{C}_t$ later in this section, we do not make use of this.

ii) In analogy to the value $p_t(X)$ for buying the asset X, we can define a value $p_t^s(X)$ for selling X by

$$\operatorname*{ess\,sup}_{g\in\mathcal{C}_t} U_t(x_t + g) = \operatorname*{ess\,sup}_{g\in\mathcal{C}_t} U_t(x_t + p_t^s(X) + g - X). \tag{6.2}$$

All results will be stated for $p_t(X)$ only, since $p_t^s(X) = -p_t(-X)$ so that the value of selling the asset X can easily be deduced from $p_t(X)$. $\diamondsuit$

Let us first consider the utility indifference value p_t for a fixed time t. Throughout this section we assume that the functional U_t is *$\mathcal{F}_t$-translation invariant* in the sense of Definition 3.1, i.e., we make the standing assumption:

Assumption (TI): The functional $U_t : \mathbf{L}^\infty \to \mathbf{L}^\infty(\mathcal{F}_t)$ satisfies

$$U_t(X + a_t) = U_t(X) + a_t \quad \text{for all } X \in \mathbf{L}^\infty \text{ and } a_t \in \mathbf{L}^\infty(\mathcal{F}_t).$$

This assumption implies (like the notation suggests) that $p_t(X)$ does not depend on the initial endowment $x_t \in \mathbf{L}^\infty(\mathcal{F}_t)$, since this can be pulled out on both sides of equation (6.1). If in addition

$$U_t^{\mathrm{opt}}(X) := \operatorname*{ess\,sup}_{g\in\mathcal{C}_t} U_t(X + g) \in \mathbf{L}^\infty \quad \text{for all } X \in \mathbf{L}^\infty, \tag{6.3}$$

then translation invariance ensures that $p_t(X)$ is well-defined in $\mathbf{L}^\infty(\mathcal{F}_t)$ and given by

$$p_t(X) = U_t^{\mathrm{opt}}(X) - U_t^{\mathrm{opt}}(0). \tag{6.4}$$

$U_t^{\mathrm{opt}}(X)$ is the maximal utility the agent can achieve from the payoff X by trading optimally in the market. It is clear from (6.4) that this operator is a key tool in the investigation of the utility indifference value.

When defining the value $p_t(X)$ as in (6.1), we implicitly assume that the agent does not yet hold any other assets due at time T. In fact, such assets might cause diversification effects which she should take into account for the valuation. Suppose the agent already holds in her portfolio an asset with payoff $Y \in \mathbf{L}^\infty$ due at time T. Then she should define $p_t^Y(X)$, the utility indifference value at time t for buying the asset X when holding Y, implicitly by

$$\operatorname*{ess\,sup}_{g \in C_t} U_t\big(x_t + g + Y\big) = \operatorname*{ess\,sup}_{g \in C_t} U_t\big(x_t - p_t^Y(X) + g + Y + X\big). \tag{6.5}$$

In other words, she should compare the maximal utility she can achieve by trading optimally when she has only Y with the maximal utility she can obtain when her portfolio consists of X and Y (and when she has to pay $p_t^Y(X)$ at time t). Analogously to (6.4), provided that U_t^{opt} maps $\mathbf{L}^\infty$ into $\mathbf{L}^\infty(\mathcal{F}_t)$, we can resolve (6.5) for $p_t^Y(X)$ to obtain

$$p_t^Y(X) = U_t^{\mathrm{opt}}(X+Y) - U_t^{\mathrm{opt}}(Y). \tag{6.6}$$

The following result shows that our approach has the pleasant property that this leads to a consistent valuation principle, in the sense that the value for $X+Y$ coincides with the sum of the value for Y and the value for X when holding Y. Put differently, it does not matter whether the agent buys the assets one after another or in bulk, always provided that she properly takes into account what has already been bought.

Proposition 6.2 *If $U_t^{\mathrm{opt}}(X) \in \mathbf{L}^\infty$ for all $X \in \mathbf{L}^\infty$ then*

$$p_t(X+Y) = p_t(Y) + p_t^Y(X).$$

Proof This follows immediately from (6.4) and (6.6). □

From now on we do not only assume that U_t is translation invariant, but that $U_t = \Phi_t$ is an MCUF at time t. Monotonicity and translation invariance of an MCUF imply that U_t^{opt} maps $\mathbf{L}^\infty$ into $\mathbf{L}^\infty(\mathcal{F}_t)$ if and only if $U_t^{\mathrm{opt}}(0)$ is bounded, i.e., if

$$U_t^{\mathrm{opt}}(0) = \operatorname*{ess\,sup}_{g \in \mathcal{C}_t} \Phi_t(g) = \Phi_t \Box \left(-\mathcal{C}_t\right)(0) \in \mathbf{L}^\infty. \tag{6.7}$$

Recall that we have studied the operator U_t^{opt} in detail in Lemma 4.5 and Proposition 4.7. In particular, we have given conditions for when it is an MCUF and also for when it corresponds to the convolution of Φ_t and

$$\Phi_t^{-\mathcal{C}_t}(X) := \operatorname{ess\,sup}\{\, m_t \in \mathbf{L}^\infty(\mathcal{F}_t) \mid X - m_t \in -\mathcal{C}_t\}, \tag{6.8}$$

the *market MCUF* induced by $\mathcal{C}_t$. This name is justified by the observation that in view of the interpretation of $\mathcal{C}_t$ as superhedgeable payoffs, $-\Phi_t^{-\mathcal{C}_t}(-X)$ is the minimal amount required at time t that allows to superhedge X. Lemma 4.5 and Proposition 4.7 together with (6.4) immediately yield the following result:

Proposition 6.3 *Let Φ_t be an MCUF at time t and $\mathcal{C}_t \subseteq \mathbf{L}^\infty$ a non-empty convex and $\mathcal{F}_t$-regular set such that (6.7) holds. Then:*

a) *$p_t(\,.\,)$ is a normalized MCUF at time t, which is continuous from below if $\Phi_t(\,.\,)$ is.*
b) *If $-\mathcal{C}_t$ is an acceptable set at time t then*

$$U_t^{\mathrm{opt}}(X) = \Phi_t \Box \Phi_t^{-\mathcal{C}_t}(X) \quad \text{for all } X \in \mathbf{L}^\infty \tag{6.9}$$

so that

$$p_t(X) = \Phi_t \Box \Phi_t^{-\mathcal{C}_t}(X) - \Phi_t \Box \Phi_t^{-\mathcal{C}_t}(0) \quad \text{for all } X \in \mathbf{L}^\infty. \tag{6.10}$$

Remark 6.4 A sufficient condition for (6.7) is that

$$\mathcal{C}_t \cap \{X \in \mathbf{L}^\infty \,|\, \mathbb{P}[\Phi_t(X) > 0] > 0\} = \emptyset, \tag{6.11}$$

since this implies that ess $\sup_{g\in\mathcal{C}_t} \Phi_t(g) \leq 0$. If Φ_t is coherent and $\mathcal{C}_t$ is a non-empty $\mathcal{F}_t$-regular convex cone containing 0, then (6.11) is even necessary for (6.7). In fact, if (6.11) does not hold, then there exists $X \in \mathcal{C}_t$ and $\epsilon > 0$ such that for $A := \{\Phi_t(X) \geq \epsilon\} \in \mathcal{F}_t$ we have $\mathbb{P}[A] > 0$. But since for all $n \in I\!N$ also $nX\mathbf{1}_A \in \mathcal{C}_t$, positive homogeneity and $\mathcal{F}_t$-regularity of Φ_t imply that

$$\operatorname*{ess\,sup}_{g\in\mathcal{C}_t} \Phi_t(g) \geq \Phi_t(nX\mathbf{1}_A) \geq n\epsilon\mathbf{1}_A.$$

Taking the limit for $n \to \infty$, this shows that (6.7) cannot hold true. ◊

It seems natural to ask if we can consider $p_t(X)$ not only as a value for X, but also as a price for (buying) X. A minimal requirement for this is clearly that $p_t(X)$ should not lead to arbitrage opportunities. To make this more precise we should first ensure that the market itself does not contain arbitrage opportunities. Therefore we impose that

$$\Phi_t^{-\mathcal{C}_t}(0) = \text{ess sup}\left(\mathcal{C}_t \cap \mathbf{L}^\infty(\mathcal{F}_t)\right) \leq 0, \tag{6.12}$$

i.e., that one cannot superhedge from t on at zero cost something known at time t and positive. In particular (6.12) ensures that the interval $[\Phi_t^{-\mathcal{C}_t}(X), -\Phi_t^{-\mathcal{C}_t}(-X)]$ from the subhedging to the superhedging price is non-empty. Then for $p_t(X)$ respectively $p_t^s(X)$ not to yield arbitrage opportunities they should lie inside the interval $(\Phi_t^{-\mathcal{C}_t}(X), -\Phi_t^{-\mathcal{C}_t}(-X))$; for an early work on this see [Fri00]. By Proposition 6.3, $p_t(X)$ is a normalized MCUF at time t, and since $p_t^s(X) = -p_t(-X)$, this implies that $p_t(X) \leq p_t^s(X)$ so that the value (or price) for buying X does not exceed the value for selling X. In fact, normalization and concavity imply that

$$0 = p_t\left(\frac{1}{2}X - \frac{1}{2}X\right) \geq \frac{1}{2}p_t(X) + \frac{1}{2}p_t(-X),$$

so that

$$-p_t(X) \geq p_t(-X)$$

on $\mathbf{L}^\infty$. Consequently, we seek for conditions which ensure that $p_t(X)$ and $p_t^s(X)$ yield arbitrage-free bid and ask prices for X in the sense that

$$[p_t(X), p_t^s(X)] \subseteq [\Phi_t^{-\mathcal{C}_t}(X), -\Phi_t^{-\mathcal{C}_t}(-X)]. \tag{6.13}$$

However, a violation of condition (6.13) does not necessarily lead to an arbitrage opportunity. Indeed, to exclude arbitrage, it would already suffice to have the two interlocking inequalities

$$p_t(X) \leq -\Phi_t^{-\mathcal{C}_t}(-X) \quad \text{and} \quad p_t^s(X) \geq \Phi_t^{-\mathcal{C}_t}(X). \tag{6.14}$$

But if for instance $p_t^s(X)$, the value for selling X, exceeds the superhedging price $-\Phi_t^{-\mathcal{C}_t}(-X)$ for buying X, nobody would agree to pay this as a price. Therefore we consider the stronger condition (6.13) to be desirable. The next result gives sufficient conditions for (6.13).

Proposition 6.5 *Let Φ_t be an MCUF at time t and $-\mathcal{C}_t \subseteq \mathbf{L}^\infty$ an acceptable set at time t such that (6.7) and (6.12) hold. Then we have absence of arbitrage in the sense of (6.13) if one of the following conditions holds:*

a) *$-\mathcal{C}_t$ is a convex cone containing 0.*
b) *0 is in the acceptance set of Φ_t and the MCUF U_t^{opt} is normalized, i.e., $\Phi_t(0) \geq 0$ and* $\underset{g \in \mathcal{C}_t}{\operatorname{ess\,sup}}\ \Phi_t(g) = 0$.

In particular, if a) or b) holds and if X satisfies $\Phi_t^{-\mathcal{C}_t}(X) = -\Phi_t^{-\mathcal{C}_t}(-X)$, then

$$\Phi_t^{-\mathcal{C}_t}(X) = p_t(X) = p_t^s(X) = -\Phi_t^{-\mathcal{C}_t}(-X).$$

Thus for an asset which is traded in the market, value and market price must coincide.

Proof Since $p_t^s(\,.\,) = -p_t(-\,.\,)$, it suffices to show that

$$\Phi_t^{-\mathcal{C}_t}(X) \leq p_t(X). \tag{6.15}$$

a) If $-\mathcal{C}_t$ is a convex cone containing 0, then $\Phi_t^{-\mathcal{C}_t}$ is by Lemma 3.8 positively homogeneous and therefore by Remark 3.2 iv) superadditive, i.e., it satisfies $\Phi_t^{-\mathcal{C}_t}(X+Y) \geq \Phi_t^{-\mathcal{C}_t}(X) + \Phi_t^{-\mathcal{C}_t}(Y)$. Hence Proposition 6.3 and the symmetry of the convolution imply that

$$\begin{aligned} p_t(X) &= \Phi_t \Box \Phi_t^{-\mathcal{C}_t}(X) - \Phi_t \Box \Phi_t^{-\mathcal{C}_t}(0) \qquad (6.16)\\ &= \underset{Y \in \mathbf{L}^\infty}{\operatorname{ess\,sup}} \Big(\Phi_t^{-\mathcal{C}_t}(X+Y) + \Phi_t(-Y) \Big) - \Phi_t \Box \Phi_t^{-\mathcal{C}_t}(0) \\ &\geq \Phi_t^{-\mathcal{C}_t}(X) + \underset{Y \in \mathbf{L}^\infty}{\operatorname{ess\,sup}} \Big(\Phi_t^{-\mathcal{C}_t}(Y) + \Phi_t(-Y) \Big) - \Phi_t \Box \Phi_t^{-\mathcal{C}_t}(0) \\ &= \Phi_t^{-\mathcal{C}_t}(X). \end{aligned}$$

b) If $\Phi_t \Box \Phi_t^{-\mathcal{C}_t}(0) = U^{\mathrm{opt}}(0) = 0$, then (6.16) simplifies to

$$\begin{aligned} p_t(X) &= \Phi_t \Box \Phi_t^{-\mathcal{C}_t}(X) \\ &= \underset{Y \in \mathbf{L}^\infty}{\operatorname{ess\,sup}} \Big(\Phi_t^{-\mathcal{C}_t}(X+Y) + \Phi_t(-Y) \Big) \\ &\geq \Phi_t^{-\mathcal{C}_t}(X) + \Phi_t(0) \\ &\geq \Phi_t^{-\mathcal{C}_t}(X), \end{aligned}$$

where the last inequality holds since $\Phi_t(0) \geq 0$.

□

When $\mathcal{C}_t$ is only convex but not a convex cone containing 0, even the weaker no-arbitrage condition (6.14) can be violated. This can be explained as follows. Our definition (6.1) of the utility indifference value uses the same set of gains $\mathcal{C}_t$ from strategies irrespective of whether the agent owns X or not, and so we implicitly assume that buying X does not change the set of possible strategies. Note that X is here viewed as a new financial *instrument*; like in a market with transaction costs, this must be distinguished from a *portfolio* generating the same payoff as X, but formed from the primary assets in the market. The following example explicitly illustrates how buying or owning such a portfolio can change the set $\mathcal{C}_0$ of allowed gains into a new set $\mathcal{C}_0^X$, and how this makes it reasonable for the agent to pay more for X than the $\mathcal{C}_0$-superhedging price. Indeed, although $p_0(X)$ is bigger than $-\Phi_0^{-\mathcal{C}_0}(-X)$, the agent cannot increase her maximal attainable utility by superhedging X via the portfolio instead of buying it directly for $p_0(X)$, because she may only work with $\mathcal{C}_0^X$ after the superhedging.

The above discussion shows that one must be very careful when introducing a new instrument X in the market, because (especially with constraints) this may affect the set of allowed trades. However, we do not pursue this delicate issue any further.

Example 6.6 For simplicity we consider a one-step discrete time model with only two possible states. There exists a bank account with zero interest rate and one risky asset S with net payoff $S_1 - S_0 = \left(-1, \frac{1}{4}\right)$. Trading is restricted in that the agent is not allowed to hold strictly less than -1 units of the risky asset. Hence the set of payoffs which can be superhedged by trading from zero initial capital is

$$\mathcal{C}_0 = \left\{ \beta \left(-1, \frac{1}{4}\right) \middle| \beta \geq -1 \right\} - I\!R^2_+.$$

We consider the payoff $X := \left(\frac{1}{2}, -\frac{1}{4}\right)$. Its superhedging price is

$$\begin{aligned} -\Phi_0^{-\mathcal{C}_0}(-X) &= \inf \left\{ c \in I\!R \,\middle|\, \left(\frac{1}{2}, -\frac{1}{4}\right) \leq c + \beta \left(-1, \frac{1}{4}\right) \text{ for some } \beta \geq -1 \right\} \\ &= \inf_{\beta \geq -1} \left\{ \max \left\{ \frac{1}{2} + \beta, -\frac{1}{4} - \frac{1}{4}\beta \right\} \right\} \\ &= -\frac{1}{10}, \end{aligned}$$

since it is easy to check that the infimum is attained for $\beta = -\frac{3}{5}$. Note that the corresponding superhedging strategy is even a hedging strategy as it perfectly replicates X. The preferences of the agent correspond to the exponential certainty equivalent from Example 3.3 with risk aversion $\frac{1}{4}$ so that

$$\Phi_0(X) = -4 \log \mathbb{E} \left[e^{-\frac{1}{4}X} \right],$$

where the probability measure $\mathbb{P}$ assigns to both possible states the same probability. Hence the maximal attainable monetary utility without owning X is

$$\begin{aligned} \sup_{g \in \mathcal{C}_0} \Phi_0(g) &= \sup_{\beta \geq -1} \Phi_0 \left(\beta \left(-1, \frac{1}{4}\right) \right) \\ &= \sup_{\beta \geq -1} \left\{ -4 \log \left\{ \frac{1}{2} e^{\frac{1}{4}\beta} + \frac{1}{2} e^{-\frac{1}{16}\beta} \right\} \right\} \\ &= -4 \log \left(\frac{1}{2} \left(e^{-\frac{1}{4}} + e^{\frac{1}{16}} \right) \right) \\ &\approx 0.3264, \end{aligned}$$

where the supremum is attained for $\beta = -1$. Along the same lines, the maximal attainable monetary utility when holding X is

$$\begin{aligned}\sup_{g\in\mathcal{C}_0} \Phi_0(X+g) &= \sup_{\beta\geq -1}\left\{-4\log\left(\frac{1}{2}e^{-\frac{1}{4}\left(\frac{1}{2}-\beta\right)}+\frac{1}{2}e^{-\frac{1}{4}\left(-\frac{1}{4}+\frac{1}{4}\beta\right)}\right)\right\} \\ &= -4\log\left(\frac{1}{2}\left(e^{-\frac{3}{8}}+e^{\frac{1}{8}}\right)\right) \\ &\approx 0.3763,\end{aligned}$$

where again the supremum is attained for $\beta = -1$. By (6.4),

$$p_0(X) = \sup_{g\in\mathcal{C}_0} \Phi_0(X+g) - \sup_{g\in\mathcal{C}_0} \Phi_0(g) \approx 0.050 > -\frac{1}{10} = -\Phi_0^{-\mathcal{C}_0}(-X) \tag{6.17}$$

so that even the weak no-arbitrage condition (6.14) is violated. Moreover, we can immediately see why this happens. In fact, the argument why prices should be consistent with the no-arbitrage principle is that instead of buying X for a price exceeding its superhedging price, it would be cheaper to buy the assets required to superhedge X. However, the situation is slightly different here. For superhedging X, the agent needs to sell short $\frac{3}{5}$ units of the risky asset, and then she can go short only $\frac{2}{5}$ further units in the risky asset. Therefore her maximal attainable monetary utility after implementing the (super-)hedging strategy for $X = -\frac{1}{10} - \frac{3}{5}\left(-1, \frac{1}{4}\right)$ is

$$\begin{aligned}\sup_{\beta\geq-\frac{2}{5}} \Phi_0\left(X - \left(-\Phi_0^{-\mathcal{C}_0}(-X)\right) + \beta\left(-1, \frac{1}{4}\right)\right) &= \sup_{\beta\geq-\frac{2}{5}} \Phi_0\left(-\frac{3}{5}\left(-1, \frac{1}{4}\right) + \beta\left(-1, \frac{1}{4}\right)\right) \\ &= U_0^{\mathrm{opt}}(0) \\ &\approx 0.3264.\end{aligned}$$

Note how the initial trade to superhedge X has explicitly changed the set of strategies from $\mathcal{C}_0 = \{\beta \geq -1\}$ to $\mathcal{C}_0^X = \{\beta \geq -\frac{2}{5}\}$. On the other hand, if directly buying X for $p_0(X)$ does not change the set of possible trading strategies, then the maximal monetary utility after that purchase is

$$\begin{aligned}\sup_{g\in\mathcal{C}_0} \Phi_0(X - p_0(X) + g) &= U_0^{\mathrm{opt}}(X) - p_0(X) \\ &= U_0^{\mathrm{opt}}(0) \\ &\approx 0.3264.\end{aligned}$$

Hence acting upon the apparent arbitrage opportunity does not yield a higher attainable utility than buying X for $p_0(X)$, since the former trade changes the set of admissible strategies. ◊

To specify the representation and the acceptance set of the convolution $\Phi_t \square \Phi_t^{-\mathcal{C}_t}$ and hence of p_t more precisely, we require some additional properties. The following result follows immediately from Proposition 4.7 (with $\mathcal{B} = -\mathcal{C}_t$) and (6.10).

Proposition 6.7 *Let $-\mathcal{C}_t$ be an acceptable set at time t and Φ_t an MCUF at time t which is continuous from below with acceptance set $\mathcal{A}_t$ and concave conjugate α_t. If (6.7) holds and if*

$$\operatorname{ess\,sup}\left\{m_t \in \mathbf{L}^\infty(\mathcal{F}_t) \mid m_t \in \overline{\mathcal{C}_t}\right\} \in \mathbf{L}^\infty,$$

where the closure is taken in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$*, then the MCUF* $U_t^{\text{opt}} = \Phi_t \Box \Phi_t^{-\mathcal{C}_t}$ *is continuous from below and its concave conjugate is*

$$\alpha_t(\mathbb{Q}) + \alpha_t^{\overline{-\mathcal{C}_t}}(\mathbb{Q}),$$

where $\alpha_t^{\overline{-\mathcal{C}_t}}(\mathbb{Q}) := \operatorname*{ess\,inf}_{Y \in \overline{-\mathcal{C}_t}} \mathbb{E}_\mathbb{Q}[Y|\mathcal{F}_t]$*. Its acceptance set is*

$$\overline{\mathcal{A}_t + \overline{-\mathcal{C}_t}} = \overline{\mathcal{A}_t - \mathcal{C}_t}.$$

In particular, the utility indifference value functional

$$p_t(\,.\,) = U_t^{\text{opt}}(\,.\,) - U_t^{\text{opt}}(0)$$

is an MCUF which is continuous from below with acceptance set

$$\overline{\mathcal{A}_t - \mathcal{C}_t} + U_t^{\text{opt}}(0).$$

Having discussed the properties of p_t for fixed t, we now investigate the dynamic aspects of the utility indifference valuation DMCUF $p = (p_t)_{0 \le t \le T}$. In particular, we turn our attention to time-consistency. Under the assumptions of Proposition 6.3 b), p_t is obtained at each time t from the convolution $\Phi_t \Box \Phi_t^{-\mathcal{C}_t}$ by normalization, and we know that normalization turns a time-consistent DMCUF into a strongly time-consistent one. We also know from Theorem 4.3 that the convolution of (strongly) time-consistent DMCUFs is again a (strongly) time-consistent DMCUF. Hence the obvious idea to ensure that the utility indifference valuation DMCUF p is strongly time-consistent is to choose Φ and the sets $(\mathcal{C}_t)_{0 \le t \le T}$ such that both Φ and the market DMCUF $(\Phi_t^{-\mathcal{C}_t})_{0 \le t \le T}$ are time-consistent. To achieve the latter by defining $\mathcal{C}_t$ in an appropriate way, we have to specify the structure of the financial market in more detail.

As in section 5 we model the discounted price process of the basic traded assets by a locally bounded RCLL $\mathbb{P}$-semimartingale $S = (S_t)_{0 \le t \le T}$. We assume that (**NFLVR**) holds and fix an admissible hedging set $\mathcal{H} \subseteq L^a_{\text{loc}}(S)$. For each time t we define the set of payoffs superhedgeable from zero initial endowment via trading during $(t, T]$ by

$$\mathcal{C}_t := \left(\left\{ \int_t^T H_s \, dS_s \;\middle|\; H \in \mathcal{H}_t \right\} - \mathbf{L}^0_+ \right) \cap \mathbf{L}^\infty \tag{6.18}$$

with

$$\mathcal{H}_t := \left\{ H \in \mathcal{H} \;\middle|\; \left(\int_t^u H_s \, dS_s \right)_{t \le u \le T} \text{ is uniformly bounded from below} \right\}. \tag{6.19}$$

Each $H \in \mathcal{H}_t$ describes a self-financing trading strategy on $(t, T]$ with a wealth process which is uniformly bounded from below. The subtraction of $\mathbf{L}^0_+$ economically means that we are always allowed to "throw away" money. In the following result we apply Theorem 5.11 to prove that the above sets $\mathcal{C}_t$ yield a strongly time-consistent market DMCUF $(\Phi_t^{-\mathcal{C}_t})_{0 \le t \le T}$.

Proposition 6.8 *For* $X \in \mathbf{L}^\infty$ *and each* $t \in [0, T]$ *define*

$$\hat{\Phi}_t(X) := -\hat{V}_t,$$

where $(\hat{V}_t)_{0\le t\le T}$ is the value process of the minimal $\mathcal{H}$-constrained hedging portfolio for $-X$ from Theorem 5.11. Then $(\hat{\Phi}_t)_{0\le t\le T}$ is a well-representable strongly time-consistent DMCUF. Its acceptance set at any time t is $-\mathcal{C}_t$ so that $\hat{\Phi}_t = \Phi_t^{-\mathcal{C}_t}$ on $\mathbf{L}^\infty$. In particular, each $\mathcal{C}_t$ is closed in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$.

Proof Clearly, $\hat{\Phi}_t(X) \in \mathbf{L}^\infty$ by uniform boundedness of $\hat{V}$. By (5.8) we can write

$$\hat{\Phi}_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}(\mathcal{S})} \left(\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] + \mathbb{E}_{\mathbb{Q}}\left[A^{\mathcal{S}}(\mathbb{Q})_T - A^{\mathcal{S}}(\mathbb{Q})_t \middle| \mathcal{F}_t \right] \right);$$

note that we construct $\hat{V}$ from $-X$. Hence Remark 3.18 i) yields that $\hat{\Phi}_t$ is indeed an MCUF at time t since in (3.8) we can set $\alpha_t^0(\mathbb{Q}) := -\mathbb{E}_{\mathbb{Q}}\left[A^{\mathcal{S}}(\mathbb{Q})_T - A^{\mathcal{S}}(\mathbb{Q})_t \middle| \mathcal{F}_t \right]$ if $\mathbb{Q} \in \mathcal{P}(\mathcal{S})$ and $\alpha_t^0(\mathbb{Q}) := -\infty$ otherwise. Now Theorem 3.16 together with Remark 3.18 ii) imply that $\hat{\Phi}_t$ is well-representable and in particular that its acceptance set $\hat{\mathcal{A}}_t$ is closed in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$. Next we show that $\hat{\mathcal{A}}_t = -\mathcal{C}_t$. To see that $-\mathcal{C}_t \subseteq \hat{\mathcal{A}}_t$, note that for any $H \in \mathcal{H}_t$ and $Y \in \mathbf{L}^0_+$ such that $g := \int_t^T H_s\, dS_s - Y \in \mathcal{C}_t$, we can construct an $\mathcal{H}$-constrained hedging portfolio $(0, H', K')$ for g by choosing $H' := H\mathbf{1}_{]\!]t,T]\!]}$ and $K' := Y\mathbf{1}_{[\![T]\!]}$, where $H' \in \mathcal{H}$ by Remark 5.3 ii). The value process V' corresponding to $(0, H', K')$ is zero at time t and, as required for an $\mathcal{H}$-constrained hedging portfolio, uniformly bounded from below since $H \in \mathcal{H}_t$ and $g \in \mathbf{L}^\infty$. This implies that the value process $\tilde{V}$ of the minimal $\mathcal{H}$-constrained hedging portfolio for g satisfies $\tilde{V}_t \le V'_t = 0$ so that $\hat{\Phi}_t(-g) = -\tilde{V}_t \ge 0$, i.e., $-g \in \hat{\mathcal{A}}_t$. To see that also $\hat{\mathcal{A}}_t \subseteq -\mathcal{C}_t$, fix $X \in \hat{\mathcal{A}}_t$ and denote by $(\hat{x}, \hat{H}, \hat{K})$ the minimal $\mathcal{H}$-constrained hedging portfolio for $-X$ and the corresponding (uniformly bounded) value process by $\hat{V}$. Since $(\hat{K}_u - \hat{K}_t)_{t\le u\le T}$ is an increasing process, we obtain from

$$\hat{V}_u = \hat{V}_t + \int_t^u \hat{H}_s\, dS_s - (\hat{K}_u - \hat{K}_t), \quad t \le u \le T \tag{6.20}$$

that $\hat{H} \in \mathcal{H}_t$. Moreover, if we take $u = T$ in (6.20) and recall that $\hat{V}_t \le 0$ (since $X \in \hat{\mathcal{A}}_t$) and $\hat{V}_T \ge -X$, this also shows that $-X \in \mathcal{C}_t$. Hence we have proved that $-\mathcal{C}_t$ is the acceptance set of $\hat{\Phi}_t$.

Since clearly $-\mathcal{C}_t \subseteq -\mathcal{C}_s$ for $t \ge s$, it only remains to show time-consistency. So let $s < t$ and suppose that $\hat{\Phi}_t(X) = \hat{\Phi}_t(Y)$, but $\mathbb{P}[\hat{\Phi}_s(X) > \hat{\Phi}_s(Y)] > 0$ for some $X, Y \in \mathbf{L}^\infty$. Denote by (x^X, H^X, K^X), (x^Y, H^Y, K^Y) the minimal $\mathcal{H}$-constrained hedging portfolios for $-X$ and $-Y$ with value processes $V^X = -\hat{\Phi}(X)$ and $V^Y = -\hat{\Phi}(Y)$. Then we can define another $\mathcal{H}$-constrained hedging portfolio (x', H', K') for $-Y$ (by essentially switching from (x^X, H^X, K^X) to (x^Y, H^Y, K^Y) at time t) via

$$\begin{aligned} x' &:= x^X, \\ H'_u &:= H_u^X \mathbf{1}_{\{u\le t\}} + H_u^Y \mathbf{1}_{\{u>t\}}, \\ K'_u &:= K_u^X \mathbf{1}_{\{u\le t\}} + (K_u^Y - K_t^Y + K_t^X)\mathbf{1}_{\{u>t\}}. \end{aligned}$$

Note that $H' \in \mathcal{H}$ by predictable convexity and that the value process corresponding to (x', H', K') is given by $V' := V^X\mathbf{1}_{[\![0,t[\![} + V^Y\mathbf{1}_{[\![t,T]\!]}$ (since $V_t^X = V_t^Y$). Hence, V' is in particular uniformly bounded (from below) so that (x', H', K') is an $\mathcal{H}$-constrained hedging portfolio for $-Y$. Since $\mathbb{P}[V'_s < V_s^Y] > 0$ we get a contradiction to the minimality of (x^Y, H^Y, K^Y). Therefore $\hat{\Phi}_s(X) = \hat{\Phi}_s(Y)$ and $\hat{\Phi}$ is time-consistent. □

Combining Proposition 6.8 and Theorem 4.3 immediately shows that we can extend Proposition 6.7 to obtain strong time-consistency as well:

Proposition 6.9 *Let Φ be a time-consistent DMCUF which is continuous from below, and such that (6.7) is satisfied for each $t \in [0,T]$ with $\mathcal{C}_t$ from (6.18). Then the utility indifference valuation DMCUF $p(\,.\,)$ is also continuous from below and strongly time-consistent.*

Remark 6.10 i) The idea for using the optional decomposition under constraints to construct an MCUF describing a financial market is due to Föllmer/Schied [FS02] in the static case; see also Section 4.8 in [FS04]. However, time-consistency aspects have apparently not been studied or proved so far.

ii) Note that (**NFLVR**) implies that $\mathcal{C}_t$ from (6.18) always satisfies the no-arbitrage condition (6.12); see also Lemma 6.15 below.

$\diamondsuit$

As mentioned in section 3, one might be interested in finding a utility indifference value $p_{s,t}(X)$ for all intermediate time horizons $t < T$ and $s \leq t$, $X \in \mathbf{L}^\infty(\mathcal{F}_t)$. This requires a definition for $U^{\mathrm{opt}}_{s,t} : \mathbf{L}^\infty(\mathcal{F}_t) \to \mathbf{L}^\infty(\mathcal{F}_s)$ so that we can set $p_{s,t}(X) := U^{\mathrm{opt}}_{s,t}(X) - U^{\mathrm{opt}}_{s,t}(0)$. We have argued in section 3 that the existence of a bank account with zero interest rate implies that we should have

$$p_{s,t}(X) = p_{s,T}(X) \quad \text{for all } X \in \mathbf{L}^\infty(\mathcal{F}_t) \tag{6.21}$$

since money can be freely transferred between t and T. (6.21) holds if and only if

$$U^{\mathrm{opt}}_{s,t}(X) - U^{\mathrm{opt}}_{s,t}(0) = U^{\mathrm{opt}}_{s,T}(X) - U^{\mathrm{opt}}_{s,T}(0) \quad \text{for all } s \leq t \leq T \text{ and all } X \in \mathbf{L}^\infty(\mathcal{F}_t), \tag{6.22}$$

and in this case, time-consistency of the family p is equivalent to its recursiveness. The natural choice $U^{\mathrm{opt}}_{s,t} := U^{\mathrm{opt}}_{s,T}(X) - U^{\mathrm{opt}}_{t,T}(0)$ satisfies (6.22) and makes sense if $(U^{\mathrm{opt}}_{s,T}(0))_{0\leq s\leq T}$ is a *deterministic* process, hence in particular if the process is constantly zero. This occurs for instance if all sets $\mathcal{C}_t$ from (6.18) are convex cones containing 0, so that the market functional is a time-consistent DMCohUF, and if in addition Φ is a time-consistent DMCohUF. Their convolution U^{opt} is then by Theorem 4.3 a strongly time-consistent DMCohUF and in particular normalized. Hence, in this coherent setting, the valuation family p is recursive as in ($\mathcal{R}$).

Remark 6.11 Let us indicate why we used the results of section 5 about superhedging under constraints to prove that $(\Phi_t^{-\mathcal{C}_t})_{0\leq t\leq T}$ is time-consistent. In Proposition 6.8, we have seen that those results imply that $-\mathcal{C}_t$ is a $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$-closed acceptable set at time t, so that it is the acceptance set of $\Phi_t^{-\mathcal{C}_t}$ and the essential supremum in the definition of $\Phi_t^{-\mathcal{C}_t}(X)$ is attained by some $m_t \in \mathbf{L}^\infty(\mathcal{F}_t)$. Moreover, Proposition 6.8 tells us that there exists an $\mathcal{H}$-constrained hedging portfolio for $-X$ such that $\Phi_t^{-\mathcal{C}_t}(X)$ corresponds at each time t to minus the value $\hat{V}_t$ of this portfolio. In particular, this value process is uniformly bounded.

Now suppose we try to find a set $\mathcal{H}$ of integrands such that $(\Phi_t^{-\mathcal{C}_t})_{0\leq t\leq T}$ becomes time-consistent, without using the results from section 5. From Lemma 3.25, we basically have two possibilities to prove time-consistency. Since the set of payoffs which can be superhedged from zero initial capital by trading during $(t,T]$ corresponds to a set of stochastic integrals with respect to the price process of the traded assets S, it seems natural to try and prove that the acceptance sets $(\mathcal{A}_t)$ of $(\Phi_t^{-\mathcal{C}_t})_{0\leq t\leq T}$ have the decomposition property

$$\mathcal{A}_s = \mathcal{A}_s(\mathcal{F}_t) + \mathcal{A}_t \quad \text{for all } s \leq t. \tag{6.23}$$

But then the following problems occur:

- It is difficult to find conditions on the set $\mathcal{H}$ of integrands allowed for trading so that $-\mathcal{C}_t$ is an acceptance set of some MCUF at each time t, e.g., conditions which ensure that $-\mathcal{C}_t$ from (6.18) is a $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$-closed acceptable set at each time t. But if this fails, the acceptance set of $\Phi_t^{-\mathcal{C}_t}$ differs from $-\mathcal{C}_t$ and we cannot expect that it has the nice structure as a set of integrals which we would like to exploit to prove (6.23). Replacing $\mathcal{C}_t$ by its closure $\overline{\mathcal{C}_t}$ in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$ at each time t we lose the above integral structure. So the difficulty here is that a closure operation in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$ does not fit well with stochastic integrals.
- Even if $-\mathcal{C}_t$ is for each time t the acceptance set of $\Phi_t^{-\mathcal{C}_t}$ and has a nice integral structure as above, we have not finished. Indeed, if $\int_s^T H\,dS$ is an element of $\mathcal{C}_s$, we can clearly split it for any $s \leq t \leq T$ into the sum of $\int_s^t H\,dS$ and $\int_t^T H\,dS$. But unfortunately, uniform boundedness from below of $(\int_s^u H\,dS)_{s\leq u\leq T}$ need not carry over to $(\int_t^u H\,dS)_{t\leq u\leq T}$, which is required if we want the latter to correspond to an element of $-\mathcal{C}_t$. So here the difficulty is to handle lower bounds on varying time intervals.

Alternatively, we might try to prove time-consistency directly from its definition, i.e., to show that

$$\Phi_t^{-\mathcal{C}_t}(X) = \Phi_t^{-\mathcal{C}_t}(Y) \quad \text{implies} \quad \Phi_s^{-\mathcal{C}_s}(X) = \Phi_s^{-\mathcal{C}_s}(Y) \tag{6.24}$$

for all $s \leq t$, where $\Phi_u^{-\mathcal{C}_u}(X) = \operatorname{ess\,sup}\{m_u \in \mathbf{L}^\infty(\mathcal{F}_u) \,|\, X - m_u \in -\mathcal{C}_u\}$. It looks natural to try this by a contradiction argument, and that involves the construction of a hedging strategy starting at time s by pasting together at time t the strategies which are associated with $\Phi_s^{-\mathcal{C}_s}(X)$ and $\Phi_t^{-\mathcal{C}_t}(Y)$. But then similar problems as above occur:

- If $-\mathcal{C}_t$ is not a $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$-closed acceptable set at time t, the supremum in the definition of $\Phi_t^{-\mathcal{C}_t}$ need not be attained. Hence we cannot relate to it one single hedging strategy, but need an entire sequence. However, pasting together countably many strategies is not feasible in general since we lose control over the required uniform lower bound for the corresponding value process.
- Even if $-\mathcal{C}_t$ is closed in $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$, so that the essential supremum is attained, it is not clear how $\Phi_s^{-\mathcal{C}_s}(X)$ and $\Phi_t^{-\mathcal{C}_t}(X)$ are related. The problem is that the value at time t of the hedging strategy corresponding to $\Phi_s^{-\mathcal{C}_s}$ need not be in $\mathbf{L}^\infty(\mathcal{F}_t)$, since it is not necessarily bounded from above.

This discussion explains why we decided to provide and work with the results about superhedging under constraints. $\diamondsuit$

We now turn to a discussion of the special case of unconstrained trading. In particular, we examine the effect of unconstrained trading on the MCUF Φ_t which expresses the preferences of an investor. For a static MCUF Φ_0, it is known (see e.g., Chapter 4.8 in [FS04] or [BEK05]) that this is captured by taking the infimum in the representation of Φ_0 only over all $\mathbb{Q} \in \mathbb{M}^a$, the set of all $\mathbb{Q} \in \mathcal{M}_1$ which are local martingale measures for S, instead of taking it over the whole set $\mathcal{M}_1$. In other words, if α_0 is the concave conjugate of Φ_0, then the new MCUF $U_0^{\text{opt}}(\,.\,) = \sup_{g \in \mathcal{C}_0} \Phi_0(\,.\,+g)$ can be represented as

$$U_0^{\text{opt}}(X) = \inf_{\mathbb{Q} \in \mathbb{M}^a} \left\{ \mathbb{E}_{\mathbb{Q}}[X] - \alpha_0(\mathbb{Q}) \right\}. \tag{6.25}$$

We shall obtain an analogous result in the dynamic case. One might expect that at time t, we have to take the essential supremum over the set of all local martingale measures for the process $(S_u)_{t \le u \le T}$, but we shall see that it is even possible to take the set of all equivalent local martingale measures for S (considered on all of $[\![0,T]\!]$).

Before we can state our result, we have to introduce some notation. *Unconstrained trading* means that we allow all admissible strategies for trading, i.e., we use the admissible hedging set $\mathcal{H} = L^a_{\text{loc}}(S)$. We denote by $L^a_t(S) := \mathcal{H}_t$ the set of all processes H in $L^a_{\text{loc}}(S)$ which are *(uniformly) admissible from time t* in the sense that the process $(\int_t^s H_u\, dS_u)_{t \le s \le T}$ is uniformly bounded from below, and by

$$\mathcal{D}_t := \left\{ \int_t^T H_u\, dS_u \,\middle|\, H \in L^a_t(S) \right\}$$

we denote the corresponding set of terminal values. Furthermore we distinguish between several sets of martingale measures:

Definition 6.12 For any $t \in [0,T]$ and $A \in \mathcal{F}_t$ we denote by $\mathbb{M}^{e,A}_t$ the set of all $\mathbb{Q} \in \mathcal{M}^e_1(\mathbb{P})$ such that $(S_s \mathbf{1}_A)_{t \le s \le T}$ is a local martingale under $\mathbb{Q}$, i.e., there exists an increasing sequence of $[t,T]$-valued stopping times τ_n with $\lim_{n\to\infty} \mathbb{P}[\tau_n < T] = 0$ such that $(S^{\tau_n}_s \mathbf{1}_A \mathbf{1}_{\{\tau_n > t\}})_{t \le s \le T}$ is a uniformly integrable martingale for each $n \in I\!N$. For $A = \Omega$ we write $\mathbb{M}^e_t := \mathbb{M}^{e,\Omega}_t$. In particular $\mathbb{M}^e := \mathbb{M}^e_0$ denotes the set of all equivalent local martingale measures for $S = (S_s)_{0 \le s \le T}$.

Theorem 6.13 *Let Φ_t be an MCUF at time t with acceptance set $\mathcal{A}_t$ and concave conjugate α_t. Assume that Φ_t is continuous from below and that* $\inf_{X \in \mathcal{A}_t} \mathbb{E}_{\mathbb{Q}}[X] > -\infty$ *for some $\mathbb{Q} \in \mathbb{M}^e$. Assume that (6.7) holds with*

$$\mathcal{C}_t = \left(\mathcal{D}_t - \mathbf{L}^0_+ \right) \cap \mathbf{L}^\infty. \tag{6.26}$$

Then we have the representation

$$U_t^{\text{opt}}(X) = \Phi_t \Box \Phi_t^{-\mathcal{C}_t}(X) = \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathbb{M}^e} \Big\{ \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q}) \Big\}. \tag{6.27}$$

Remark 6.14 Both (6.7) and the assumption that $\inf_{X \in \mathcal{A}_t} \mathbb{E}_{\mathbb{Q}}[X] > -\infty$ for some $\mathbb{Q} \in \mathbb{M}^e$ formalize the intuitive requirement that the a priori preferences Φ_t should fit together with the financial market. Like in the comment after Lemma 3.25, the second condition (involving $\mathbb{Q}$) need only hold for $t = 0$ if Φ is strongly time-consistent. $\Diamond$

In order to prove Theorem 6.13, we need to characterize the set $\mathbb{M}^e$ of equivalent local martingale measures in terms of $\mathcal{D}_t$.

Lemma 6.15 *Let $t \in [0,T]$, $A \in \mathcal{F}_t$, $\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P})$. Then*

$$\begin{aligned} \mathbb{Q} \in \mathbb{M}_t^{e,A} \quad &\Longleftrightarrow \quad \mathbb{E}_{\mathbb{Q}}[g\mathbf{1}_A|\mathcal{F}_t] \le 0 \quad \mathbb{Q}\text{- a.s. for all } g \in \mathcal{D}_t \cap \mathbf{L}^\infty \\ &\Longleftrightarrow \quad \mathbb{E}_{\mathbb{Q}}[g\mathbf{1}_A] \le 0 \quad \text{for all } g \in \mathcal{D}_t \cap \mathbf{L}^\infty. \end{aligned}$$

Proof The second equivalence is trivial since $\mathcal{D}_t$ is closed under multiplication with $\mathbf{1}_B$, $B \in \mathcal{F}_t$. Hence we only have to prove the first equivalence.

"$\Rightarrow$": Let $\mathbb{Q} \in \mathbb{M}_t^{e,A}$. Then $(S_s\mathbf{1}_A)_{t\le s\le T}$ is a local $\mathbb{Q}$-martingale. Each $g \in \mathcal{D}_t$ satisfies $g = \int_t^T H_s\, dS_s$ for some $H \in L_t^a(S)$. By Corollary 3.5 of [AS94], the uniform boundedness from below of $(\mathbf{1}_A \int_t^s H_s\, dS_s)_{t\le s\le T}$ implies that $(\mathbf{1}_A \int_t^s H_s\, dS_s)_{t\le s\le T}$ is also a local $\mathbb{Q}$-martingale and therefore, again by uniform boundedness from below, a $\mathbb{Q}$-supermartingale. Hence $\mathbb{E}_{\mathbb{Q}}[g\mathbf{1}_A|\mathcal{F}_t] \le 0 \quad \mathbb{Q}$- a.s.

"$\Leftarrow$": Since $(S_s\mathbf{1}_A)_{t\le s\le T}$ is locally bounded, it is a local $\mathbb{Q}$-martingale if and only if $(S_s^\tau \mathbf{1}_A \mathbf{1}_{\{\tau>t\}})_{t\le s\le T}$ is a $\mathbb{Q}$-martingale for each stopping time $t \le \tau \le T$, such that $(S_s^\tau \mathbf{1}_A \mathbf{1}_{\{\tau>t\}})_{t\le s\le T}$ is uniformly bounded. For $t \le s_1 \le s_2 \le T$ and $B \in \mathcal{F}_{s_1}$, define $H := \mathbf{1}_{]\!]\tau\wedge s_1, \tau\wedge s_2]\!]}\mathbf{1}_B$ which is in $L_t^a(S)$. By assumption,

$$\begin{aligned} 0 \ge \mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}_A \int_t^T H_s\, dS_s\right] \quad &= \quad \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_B(\mathbf{1}_A S_{s_2}^\tau - \mathbf{1}_A S_{s_1}^\tau)] \\ &= \quad \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_B(\mathbf{1}_A S_{s_2}^\tau \mathbf{1}_{\{\tau>t\}} - \mathbf{1}_A S_{s_1}^\tau \mathbf{1}_{\{\tau>t\}})], \end{aligned}$$

and since $B \in \mathcal{F}_{s_1}$ is arbitrary, we get $\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_A S_{s_2}^\tau \mathbf{1}_{\{\tau>t\}}|\mathcal{F}_{s_1}] \le \mathbf{1}_A S_{s_1}^\tau \mathbf{1}_{\{\tau>t\}}$ $\mathbb{Q}$- a.s. Because we also have $-H \in L_t^a(S)$, we even get equality, and so $(S_s^\tau \mathbf{1}_A \mathbf{1}_{\{\tau>t\}})_{t\le s\le T}$ is a $\mathbb{Q}$-martingale.

$\square$

Proof of Theorem 6.13

1) By Proposition 6.8, $\Phi_t^{-\mathcal{C}_t}$ is a well-representable MCUF at time t with acceptance set $-\mathcal{C}_t$. Because $\mathcal{C}_t$ is a convex cone containing 0, the concave conjugate $\alpha_t^{-\mathcal{C}_t}$ of $\Phi_t^{-\mathcal{C}_t}$ only takes the values 0 and $-\infty$. We claim that we have for each $\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P})$ the explicit expression

$$\alpha_t^{-\mathcal{C}_t}(\mathbb{Q}) = -\infty\, \mathbf{1}_{(A^{\mathbb{Q}})^c}, \tag{6.28}$$

where we define $A^{\mathbb{Q}} \in \mathcal{F}_t$ up to nullsets by

$$\mathbf{1}_{A^{\mathbb{Q}}} = \text{ess sup}\left\{\mathbf{1}_A \,\middle|\, A \in \mathcal{F}_t \text{ and } \mathbb{Q} \in \mathbb{M}_t^{e,A}\right\}.$$

Intuitively, $A^{\mathbb{Q}}$ is the largest $\mathcal{F}_t$-measurable set on which $(S_s)_{t\le s\le T}$ is a local $\mathbb{Q}$-martingale. To see (6.28), note first that $\mathbb{Q} \in \mathbb{M}_t^{e,A^{\mathbb{Q}}}$. Since $0 \in \mathcal{C}_t$ and $\mathcal{D}_t \cap \mathbf{L}^\infty \subseteq \mathcal{C}_t$, Lemma 6.15 implies that

$$\mathbf{1}_{A^{\mathbb{Q}}} \operatorname*{ess\,inf}_{g\in -\mathcal{C}_t} \mathbb{E}_{\mathbb{Q}}[g|\mathcal{F}_t] \equiv 0 \quad \mathbb{P}\text{- a.s.},$$

which means by Lemma 3.12 that $\alpha_t^{-\mathcal{C}_t}(\mathbb{Q}) = 0$ on $A^{\mathbb{Q}}$. To prove (6.28), it thus only remains to show that

$$\underset{g \in -\mathcal{C}_t}{\operatorname{ess\,inf}}\ \mathbb{E}_{\mathbb{Q}}[g|\mathcal{F}_t] = -\infty \quad \mathbb{P}\text{- a.s. on } (A^{\mathbb{Q}})^c.$$

For this, we may assume that $\mathbb{P}[(A^{\mathbb{Q}})^c] > 0$ so that $(S_s)_{t \le s \le T}$ with positive probability fails to be a local $\mathbb{Q}$-martingale. By Lemma 6.15, we can thus find a set $B \in \mathcal{F}_t$ with $\mathbb{P}[B] > 0$ and $B \subseteq (A^{\mathbb{Q}})^c$ and some $g_0 \in \mathcal{D}_t \cap \mathbf{L}^\infty \subseteq \mathcal{C}_t$ such that $\mathbb{E}_{\mathbb{Q}}[-g_0|\mathcal{F}_t] \le -\epsilon$ on B for some $\epsilon > 0$. Closedness of $\mathcal{C}_t$ under multiplication with non-negative scalars then implies that $\underset{g \in -\mathcal{C}_t}{\operatorname{ess\,inf}}\ \mathbb{E}_{\mathbb{Q}}[g|\mathcal{F}_t] = -\infty$ on B. But this must even hold on the whole set $(A^{\mathbb{Q}})^c$. In fact, if it does not, we obtain some set $\tilde{B} \in \mathcal{F}_t$ with $\mathbb{P}[\tilde{B}] > 0$ and $\tilde{B} \subseteq (A^{\mathbb{Q}})^c$ such that $0 \ge \underset{g \in -\mathcal{C}_t}{\operatorname{ess\,inf}}\ \mathbb{E}_{\mathbb{Q}}[g|\mathcal{F}_t] \ge -m > -\infty$ on $\tilde{B}$ for some $m > 0$. Closedness of $\mathcal{C}_t$ under multiplication with non-negative scalars now implies that $\underset{g \in -\mathcal{C}_t}{\operatorname{ess\,inf}}\ \mathbb{E}_{\mathbb{Q}}[g|\mathcal{F}_t] = 0$ on $\tilde{B}$ and therefore by Lemma 6.15 that $\mathbb{Q} \in \mathbb{M}_t^{e, A^{\mathbb{Q}} \cup \tilde{B}}$. But this contradicts the definition of $A^{\mathbb{Q}}$, and hence we have proved (6.28).

2) From (6.28), our assumptions and Theorem 4.3, $\Phi_t \Box \Phi_t^{-\mathcal{C}_t}$ is well-representable and since convoluting two MCUFs means adding their concave conjugates we obtain

$$\Phi_t \Box \Phi_t^{-\mathcal{C}_t}(X) = \underset{\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P})}{\operatorname{ess\,inf}} \Big\{ \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q}) + \infty\, \mathbf{1}_{(A^{\mathbb{Q}})^c} \Big\}. \tag{6.29}$$

This suggests that it should be enough to take the above essential infimum only over those $\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P})$ that have $\mathbb{P}[A^{\mathbb{Q}}] = 1$, which means that $\mathbb{Q}$ should be in $\mathbb{M}_t^e$. We now prove that this is true by showing that

$$\Phi_t \Box \Phi_t^{-\mathcal{C}_t}(X) = \underset{\mathbb{Q}' \in \mathbb{M}_t^e}{\operatorname{ess\,inf}} \left\{ \mathbb{E}_{\mathbb{Q}'}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q}') \right\}. \tag{6.30}$$

By (**NFLVR**), there exists $\hat{\mathbb{Q}} \in \mathbb{M}^e \subseteq \mathbb{M}_t^e$ with density process $\hat{Z}$. For any $\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P})$ with density process $Z^{\mathbb{Q}}$, define a new measure $\mathbb{Q}' \in \mathcal{M}_1^e(\mathbb{P})$ with density process Z' by

$$\frac{d\mathbb{Q}'}{d\mathbb{P}} := \mathbf{1}_{A^{\mathbb{Q}}} \frac{Z_T^{\mathbb{Q}}}{Z_t^{\mathbb{Q}}} + \mathbf{1}_{(A^{\mathbb{Q}})^c} \frac{\hat{Z}_T}{\hat{Z}_t}$$

so that $\mathbb{Q}' \in \mathbb{M}_t^e$ by the definition of $A^{\mathbb{Q}}$. Since

$$\mathbb{E}_{\mathbb{Q}'}[\,.\,|\mathcal{F}_t] = \mathbf{1}_{A^{\mathbb{Q}}}\, \mathbb{E}_{\mathbb{Q}}[\,.\,|\mathcal{F}_t] + \mathbf{1}_{(A^{\mathbb{Q}})^c}\, \mathbb{E}_{\hat{\mathbb{Q}}}[\,.\,|\mathcal{F}_t]$$

we obtain from Lemma 3.12 and (6.28) that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}'}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q}') - \alpha_t^{-\mathcal{C}_t}(\mathbb{Q}') \\
&= \mathbf{1}_{A^{\mathbb{Q}}}\Big(\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q}) - \alpha_t^{-\mathcal{C}_t}(\mathbb{Q})\Big) \\
&\qquad + \mathbf{1}_{(A^{\mathbb{Q}})^c}\Big(\mathbb{E}_{\hat{\mathbb{Q}}}[X|\mathcal{F}_t] - \alpha_t(\hat{\mathbb{Q}}) - \alpha_t^{-\mathcal{C}_t}(\hat{\mathbb{Q}})\Big) \\
&= \mathbf{1}_{A^{\mathbb{Q}}}\Big(\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q}) + \infty\,\mathbf{1}_{(A^{\mathbb{Q}})^c}\Big) \\
&\qquad + \mathbf{1}_{(A^{\mathbb{Q}})^c}\Big(\mathbb{E}_{\hat{\mathbb{Q}}}[X|\mathcal{F}_t] - \alpha_t(\hat{\mathbb{Q}}) + \infty\,\mathbf{1}_{(A^{\hat{\mathbb{Q}}})^c}\Big).
\end{aligned}$$

But $(A^{\hat{\mathbb{Q}}})^c$ is a $\mathbb{P}$-nullset since $\hat{\mathbb{Q}} \in \mathbb{M}_t^e$ and so $\alpha_t^{-\mathcal{C}_t}(\hat{\mathbb{Q}}) = 0 = -\infty\mathbf{1}_{(A^{\hat{\mathbb{Q}}})^c}$ by (6.28). The same is true for $\mathbb{Q}'$. Hence, using $A^{\mathbb{Q}} \cap (A^{\mathbb{Q}})^c = \emptyset$ and (6.28) for $\mathbb{Q}'$, we get

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}'}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q}') \\
&= \mathbb{E}_{\mathbb{Q}'}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q}') + \infty\,\mathbf{1}_{(A^{\mathbb{Q}'})^c} \\
&= \mathbf{1}_{A^{\mathbb{Q}}}\Big(\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q})\Big) + \mathbf{1}_{(A^{\mathbb{Q}})^c}\Big(\mathbb{E}_{\hat{\mathbb{Q}}}[X|\mathcal{F}_t] - \alpha_t(\hat{\mathbb{Q}})\Big) \\
&\le \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q}) + \infty\,\mathbf{1}_{(A^{\mathbb{Q}})^c}
\end{aligned}$$

by looking separately at $A^{\mathbb{Q}}$ and $(A^{\mathbb{Q}})^c$. This shows that we can replace any $\mathbb{Q} \in \mathcal{M}_1^e(\mathbb{P})$ by a corresponding $\mathbb{Q}' \in \mathbb{M}_t^e$ when taking the essential infimum in (6.29) and thus establishes (6.30).

3) In view of (6.30), it only remains to show that

$$\operatorname*{ess\,inf}_{\mathbb{Q}\in\mathbb{M}_t^e}\big\{\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q})\big\} = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathbb{M}^e}\big\{\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q})\big\}.$$

The inequality "$\le$" is clear since $\mathbb{M}_t^e \supseteq \mathbb{M}^e$. To prove the converse, we show that for any $\mathbb{Q} \in \mathbb{M}_t^e$ with density process Z, there exists $\mathbb{Q}' \in \mathbb{M}^e$ with density process Z' such that

$$Z_T = h_t Z'_T$$

for some $\mathcal{F}_t$-measurable $h_t > 0$. Because then we have from $\frac{Z_T}{Z_t} = \frac{Z'_T}{Z'_t}$ and by using (3.5) that

$$\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q}) = \mathbb{E}_{\mathbb{Q}'}[X|\mathcal{F}_t] - \alpha_t(\mathbb{Q}'),$$

and obtain "$\ge$". To construct $\mathbb{Q}'$, take some $\hat{\mathbb{Q}} \in \mathbb{M}^e$ with density process $\hat{Z}$ and define

$$Z'_T := \hat{Z}_t \frac{Z_T}{Z_t} = \frac{1}{h_t} Z_T$$

with $h_t = \frac{Z_t}{\hat{Z}_t}$. Then $\mathbb{Q}' \in \mathbb{M}^e$ because $Z'S$ is a local $\mathbb{P}$-martingale on all of $[\![0,T]\!]$: on $[\![0,t]\!]$ because $Z' = \hat{Z}$ on $[\![0,t]\!]$ and $\hat{\mathbb{Q}} \in \mathbb{M}^e$, and on $[\![t,T]\!]$ because

$$Z' = \frac{1}{h_t} Z \quad \text{on } [\![t,T]\!]$$

and ZS is a local $\mathbb{P}$-martingale on $[\![t,T]\!]$ since $\mathbb{Q} \in \mathbb{M}_t^e$. This completes the proof.

□

As an immediate consequence we get the following no-arbitrage result for the utility indifference value in the case of unconstrained trading:

Corollary 6.16 *Under the assumptions of Theorem 6.13 and with $\mathcal{C}_t$ as in (6.26), the valuations p_t and p_t^s are consistent with the no-arbitrage principle in the following two senses:*

a) *If $X \in \mathbf{L}^\infty$ is attainable from time t in the sense that $X = x_t + \int_t^T H_s\,dS_s$ with $x_t \in \mathbf{L}^\infty(\mathcal{F}_t)$ and $H \in L_t^a(S)$ such that $\left(\int_t^u H_s\,dS_s\right)_{t\le u\le T}$ is uniformly bounded, then*
$$p_t(X) = p_t\left(x_t + \int_t^T H_s\,dS_s\right) = p_t^s\left(x_t + \int_t^T H_s\,dS_s\right) = p_t^s(X) = x_t \quad \mathbb{P}\text{- a.s.}$$

b) *Both, p_t and p_t^s, take values in the interval of possible arbitrage-free valuations, i.e.,*
$$\operatorname*{ess\,inf}_{\mathbb{Q}\in\mathbb{M}^e} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] \le p_t(X) \le p_t^s(X) \le \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathbb{M}^e} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] \quad \text{for all } X \in \mathbf{L}^\infty.$$

Proof a) Since we have $\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] = x_t$ for any $\mathbb{Q} \in \mathbb{M}^e$, this follows immediately from (6.10) and the representation (6.27).

b) Since $-\mathcal{C}_t$ is a convex cone containing 0, this follows from Proposition 6.5 and Remark 5.12.

□

In all of section 6, we assumed that it is the MCUF Φ representing the agent's preferences which is continuous from below, and not the market MCUF $\Phi_t^{-\mathcal{C}_t}$. (Note that by Remark 4.2 i), for Theorem 4.3 it is enough if one of the two is continuous from below.) The reason is the following. It is known that in the unconstrained case we can represent $\Phi_0^{-C_0}$ analogously to (6.25) as
$$\Phi_0^{-C_0}(X) = \inf_{\mathbb{Q}\in\mathbb{M}^a} \mathbb{E}_{\mathbb{Q}}[X],$$
where $\mathbb{M}^a$ denotes the set of all local martingale measures $\mathbb{Q} \in \mathcal{M}_1(\mathbb{P})$ for S. It follows from Corollary 4.35 of [FS04] that continuity from below of $\Phi_0^{-C_0}$ implies that $\mathbb{M}^a$ is weakly compact (since it is weakly closed). But if the price process S is continuous and the filtration is quasi left-continuous, Corollary 7.2 of [Del92] then implies that $\mathbb{M}^a$ is a singleton so that the market must be complete. This shows that it may be rather restrictive to insist on a market DMCUF which is continuous from below.

We finish this section with a comment about the connection between the utility indifference values $p_t(X)$, $p_t^s(X)$ and *good deal bounds*.

The no-arbitrage price bounds $\Phi_t^{-\mathcal{C}_t}(\,.\,)$ and $-\Phi_t^{-\mathcal{C}_t}(-\,.\,)$ induced by superhedging are usually not sharp enough to be useful in practice of pricing. Therefore several approaches have been suggested to define tighter price bounds which are less restrictive than the choice of one pricing measure; see, e.g., [BL00], [CSR00] or [CGM01]. In particular, Cochrane and Saà-Requejo [CSR00] introduced the concept of *good deal bounds*. These price bounds are obtained by ruling out not only arbitrage opportunities but also *good deals*, which are in [CSR00] defined as investment opportunities with a high Sharpe ratio.

This procedure is justified by arguing that Sharpe ratios observed in the market tend to be rather low. Subsequently, the good deal pricing approach has been generalized by many authors; see, e.g., [JK01], [CH02], [Cer03] or [Sta04]. In particular, they defined good deals more generally as investment opportunities which are in some sense desirable and do not just have a high Sharpe ratio. To justify the exclusion of good deals, it is argued like for arbitrage opportunities that they would vanish immediately from the market by trading.

For these good deal price bounds, it is well known that they correspond to risk measures (and hence to MCUFs). However, the literature often creates the impression that they are somehow generic and independent of individual preferences. This is not the case: One has to specify the set of good deals, and we shall see presently that this basically corresponds to the choice of an MCUF and hence of a specification of utility.

The following definition of good deals (in a static and coherent framework) is taken from Jaschke and Küchler [JK01]. They fix $\mathcal{C}_0$, a convex cone containing zero of payoffs which can be superhedged with zero initial capital, and a coherent acceptance set $\mathcal{A}_0 \subseteq \mathbf{L}^\infty$, i.e., $\mathcal{A}_0$ is the acceptance set of some MCohUF at time 0. This specifies the set of acceptable payoffs and the most conservative choice is $\mathcal{A}_0 = \mathbf{L}^\infty_+$. In this case, the good deal price bounds correspond to those obtained by excluding arbitrage opportunities only.

Definition 6.17 An element $X \in \mathcal{C}_0$ is called *good deal of the first kind* if $X \in \mathcal{A}_0$ and $X \neq 0$, and *good deal (of the second kind)* if there exists $\epsilon > 0$ such that $X - \epsilon \mathbf{1}_\Omega \in \mathcal{A}_0$.

Whereas good deals of the first kind represent opportunities to get something good for free, where the good part may or may not come, those of the second kind are "cash-and-carry good deals" and yield a sure profit. Jaschke and Küchler consider the second concept to be much more important. They argue that any arbitrage transaction in practice involves some risks or costs that cannot be captured in a model. Therefore arbitrageurs will only act if the anticipated gain is substantial enough. As a consequence, they only consider good deals of the second kind, and we do the same here. The lower bound for prices for X obtained by excluding these good deals is given by

$$\pi_0^\ell(X) := \sup\{\, m_0 \in I\!R \,|\, X - m_0 \mathbf{1}_\Omega + g \in \mathcal{A}_0 \quad \text{for some } g \in \mathcal{C}_0 \}\,.$$

In fact, if the agent could buy the future payoff X for a price $\pi_0(X) < \pi_0^\ell(X)$, then there exist $g \in \mathcal{C}_0$ and $\epsilon > 0$ with $\pi_0(X) + \epsilon \leq \pi_0^\ell(X) - \epsilon$ and such that $X - (\pi_0(X) + \epsilon)\mathbf{1}_\Omega + g$ is contained in the acceptable set $\mathcal{A}_0$. Hence the agent could buy X for $\pi_0(X)$, use the superhedging strategy corresponding to g and obtain a resulting payoff $X - \pi_0(X) + g$ which is a good deal. As before, selling X corresponds to buying $-X$, and so the good deal price bounds are given by

$$[\pi_0^\ell(X), -\pi_0^\ell(-X)].$$

The above concept of good deal price bounds can immediately be generalized to a dynamic and convex framework. For a convex (but still static) setting this can also be found in Staum [Sta04]. However, he works with a slightly different definition, and the one given in [JK01] fits better into our framework. We model the set $\mathcal{C}_t$ of payoffs which are superhedgeable via trading during $(t, T]$ by a non-empty, convex and $\mathcal{F}_t$-regular subset of $\mathbf{L}^\infty$; compare Lemma 4.5. The set of acceptable payoffs is given by some acceptable set $\mathcal{B}_t$ at time t. In analogy to the static case, we then define a good deal as follows:

Definition 6.18 Fix $Y \in \mathcal{B}_t$. Then $X \in \mathcal{C}_t$ is called a *good deal at time* t if there exists a constant $\epsilon > 0$ and a set $A \in \mathcal{F}_t$ with $\mathbb{P}[A] > 0$ such that $(X - \epsilon \mathbf{1}_\Omega)\, \mathbf{1}_A + Y \mathbf{1}_{A^c} \in \mathcal{B}_t$.

Note that $\mathcal{B}_t$ is $\mathcal{F}_t$-regular so that the definition does not depend on the choice of the element $Y \in \mathcal{B}_t$; this is introduced since whether X is a good deal at time t or not should not depend on events which can already be ruled out at this time. Note that also $\mathcal{B}_t$ need not contain 0 which is otherwise a natural choice for Y. The lower price bound obtained from excluding good deals is then given by

$$\pi_t^\ell(X) := \text{ess sup} \{ m_t \in \mathbf{L}^\infty(\mathcal{F}_t) \,|\, X - m_t + g \in \mathcal{B}_t \quad \text{for some } g \in \mathcal{C}_t \} \,. \tag{6.31}$$

The reasoning is similar to the static case. Indeed, if the agent could buy the future payoff X for a price $\pi_t(X)$ which is not greater or equal to $\pi_t^\ell(X)$, then there exist $\epsilon > 0$ and a set $A \in \mathcal{F}_t$ with $\mathbb{P}[A] > 0$ such that $\pi_t(X) + \epsilon \mathbf{1}_\Omega \le \pi_t^\ell(X) - \epsilon \mathbf{1}_\Omega$ on A. By (6.31) we can find a subset $B \in \mathcal{F}_t$ of A with $\mathbb{P}[B] > 0$, $m_t \in \mathbf{L}^\infty(\mathcal{F}_t)$ and $g \in \mathcal{C}_t$ such that $Y' := X - m_t + g \in \mathcal{B}_t$ and $m_t \ge \pi_t^\ell(X) - \epsilon \mathbf{1}_\Omega$ on B. The $\mathcal{F}_t$-regularity of $\mathcal{B}_t$ implies that also $Y' \mathbf{1}_B + Y \mathbf{1}_{B^c} \in \mathcal{B}_t$. But since $\pi_t(X) + \epsilon \mathbf{1}_\Omega \le m_t$ on B and $\mathcal{B}_t$ is solid, we now obtain that $((X - \pi_t(X) + g) - \epsilon \mathbf{1}_\Omega)\, \mathbf{1}_B + Y \mathbf{1}_{B^c} \in \mathcal{B}_t$, i.e., that $X - \pi_t(X) + g$ is a good deal.

Next we show how the above price bound is connected to a utility indifference valuation functional $p_t(X)$. To this end, we recall from (3.3) in Lemma 3.8 that $\mathcal{B}_t$ induces an MCUF U_t at time t by

$$\Phi_t(X) := \Phi_t^{\mathcal{B}_t}(X) = \text{ess sup} \{ m_t \in \mathbf{L}^\infty(\mathcal{F}_t) |\, X - m_t \in \mathcal{B}_t \} \,.$$

This representation implies that

$$\begin{aligned} U_t^{\text{opt}}(X) &= \operatorname*{ess\,sup}_{g \in \mathcal{C}_t} U_t(X + g) \\ &= \operatorname*{ess\,sup}_{g \in \mathcal{C}_t} \ \text{ess sup} \{ m_t \in \mathbf{L}^\infty(\mathcal{F}_t) |\, X + g - m_t \in \mathcal{B}_t \} \\ &= \text{ess sup} \{ m_t \in \mathbf{L}^\infty(\mathcal{F}_t) \,|\, X - m_t + g \in \mathcal{B}_t \quad \text{for some } g \in \mathcal{C}_t \} \\ &= \pi_t^\ell(X). \end{aligned}$$

Hence if $U_t^{\text{opt}}(0) = 0$ so that $p_t(X) = U_t^{\text{opt}}(X)$, the lower good deal bound is the utility indifference value $p_t(X)$ and

$$[p_t(X), p_t^s(X)]$$

is the interval of possible prices for X which do not yield a good deal. We recall from Proposition 6.5 that we might need additional assumptions to have price bounds which are actually tighter than those obtained by excluding arbitrage opportunities.

Using that $p_t(\,.\,)$ is defined as the utility indifference value, we can also give another interpretation for why $p_t(\,.\,)$ can be viewed as a lower price bound obtained by excluding (slightly differently defined) good deals. We fix a set $\mathcal{C}_t$ of superhedgeable payoffs and an MCUF U_t. Then we might call $X \in \mathcal{C}_t$ *useful deal* if it increases the maximal attainable utility, i.e., if

$$U_t^{\text{opt}}(X) = \operatorname*{ess\,sup}_{g \in \mathcal{C}_t} U_t(X + g) \ge \operatorname*{ess\,sup}_{g \in \mathcal{C}_t} U_t(g) = U_t^{\text{opt}}(0)$$

and the inequality is strict with strictly positive probability. This implies that

$$[p_t(X),\, p_t^s(X)]$$

is the interval of all prices for X which do not yield a useful deal.

Staum [Sta04] proves fundamental theorems of asset pricing for good deal bounds. In particular, he gives in his Theorem 6.1 an equivalent condition for the weak no-arbitrage condition (6.14). This theorem and its proof can easily be adapted to our framework; we simply state the result without giving a proof.

Theorem 6.19 *Let $-\mathcal{C}_t \subseteq \mathbf{L}^\infty$ be an acceptable set at time t containing 0 such that $\Phi_t^{-\mathcal{C}_t}(0) = 0$. Let U_t be an MCUF at time t with acceptance set $\mathcal{A}_t$ such that $U_t(0) \geq 0$. Then*

$$p_t(X) \leq -\Phi_t^{-\mathcal{C}_t}(-X) \text{ for all } X \in \mathbf{L}^\infty \quad \text{and} \quad U_t^{\mathrm{opt}}(0) = \operatorname*{ess\,sup}_{g \in \mathcal{C}_t} U_t(g) = 0$$

if and only if

$$(\mathcal{C}_t - \mathcal{A}_t) \cap \left\{ X \in \mathbf{L}^\infty \;\middle|\; \mathbb{P}\left[\Phi_t^{-\mathcal{C}_t}(X) > 0\right] > 0 \right\} = \emptyset.$$

7 Examples

7.1 DMCUFs, Utility Indifference and BSDEs.

In this subsection we first recall and extend some known results about DMCUFs which are described by backward stochastic differential equations (BSDEs for short), since this provides us with a big class of time-consistent DMCUFs. Then we represent the preferences of our investor by such a DMCUF Φ and try to express the corresponding utility indifference valuation DMCUF in terms of BSDEs as well. As in section 6, we apply the convolution to Φ and the market DMCUF given via the superhedging price to obtain an equivalent description for the utility indifference value respectively for the DMCUF U^{opt}. To this end, we first prove that the market DMCUF can also be described by a BSDE. Then we show that the DMCUF U^{opt} corresponds to a BSDE whose driver is given by the pointwise convolution of the drivers for Φ and for the market DMCUF. This extends results of Barrieu and El Karoui [BEK04] about the convolution of dynamic risk measures described by BSDEs.

We start by recalling a well-known existence result for solutions of BSDEs. To this end we introduce some notation and conventions. In particular, we require a very special structure of the filtration since the proof of the existence result relies on a martingale representation theorem.

Remark 7.1 An existence proof based on fixed point arguments instead of a martingale representation theorem can be found in [EKH97]. However, the integrability conditions there are too restrictive for our purposes. ◊

Let $W = (W_t)_{0 \leq t \leq T}$ be a standard d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ be the augmented filtration generated by W. As before, we assume that $\mathcal{F} = \mathcal{F}_T$. We introduce the notation $\mathbf{M}_{\mathbb{F}}^2(0, T; \mathbb{R}^n)$ for the space of all equivalence classes of $\mathbb{R}^n$-valued, $\mathbb{F}$-progressively measurable processes $(\vartheta_t)_{0 \leq t \leq T}$

such that

$$\mathbb{E}\left[\int_0^T \|\vartheta_t\|^2\,dt\right] < \infty,$$

where $\|\,.\,\|$ stands for the Euclidean norm. Hence two processes ϑ^1 and ϑ^2 are identified in $\mathbf{M}^2_{\mathbb{F}}(0,T;I\!\!R^n)$ if

$$\mathbb{E}\left[\int_0^T \|\vartheta^1_t - \vartheta^2_t\|^2\,dt\right] = 0.$$

The drivers which appear in the BSDEs we consider are product-measurable functions $g : \Omega \times [0,T] \times I\!\!R \times I\!\!R^d \to I\!\!R$. We often write $g_t(y,z)$ instead of $g(\omega,t,y,z)$ and usually impose some of the following properties:

Definition 7.2 $(\mathcal{A})$ $(\omega,t) \mapsto g(\omega,t,y,z)$ is in $\mathbf{M}^2_{\mathbb{F}}(0,T;I\!\!R)$ for any $y \in I\!\!R$, $z \in I\!\!R^d$.

$(\mathcal{B})$ g is Lipschitz in $(y,z) \in I\!\!R \times I\!\!R^d$, i.e., there exists a constant $C > 0$ such that $d\mathbb{P} \otimes dt$-a.s. for all $(y_0,z_0), (y_1,z_1) \in I\!\!R \times I\!\!R^d$

$$|g_t(y_0,z_0) - g_t(y_1,z_1)| \le C(|y_0 - y_1| + \|z_0 - z_1\|).$$

$(\mathcal{C})$ $d\mathbb{P} \otimes dt$-a.s., g satisfies $g_t(y,0) \equiv 0$ for any $y \in I\!\!R$.

$(\mathcal{D})$ g does not depend on y.

$(\mathcal{E})$ g is concave in (y,z), i.e. $d\mathbb{P}\otimes dt$-a.s. for all (y_0,z_0), $(y_1,z_1) \in I\!\!R\times I\!\!R^d$ and $\alpha \in (0,1)$

$$g_t\big(\alpha y_0 + (1-\alpha)y_1, \alpha z_0 + (1-\alpha)z_1\big) \ge \alpha g_t(y_0,z_0) + (1-\alpha) g_t(y_1,z_1).$$

$(\mathcal{F})$ g is positively homogeneous in (y,z), i.e., $d\mathbb{P} \otimes dt$-a.s. for all $\lambda \ge 0$ and $(y,z) \in I\!\!R \times I\!\!R^d$

$$g_t(\lambda y, \lambda z) = \lambda g_t(y,z).$$

The following result is taken from Peng [Pen97], Proposition 36.4; see also Pardoux and Peng [PP90], Theorem 4.1.

Theorem 7.3 *Let g satisfy $(\mathcal{A})$ and $(\mathcal{B})$ of Definition 7.2. For any (fixed) $X \in \mathbf{L}^2 = \mathbf{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$ there exists a unique pair of processes $(y,z) \in \mathbf{M}^2_{\mathbb{F}}(0,T;I\!\!R) \times \mathbf{M}^2_{\mathbb{F}}(0,T;I\!\!R^d)$ with y continuous, satisfying the BSDE*

$$y_t = X + \int_t^T g_s(y_s,z_s)\,ds - \int_t^T z^*_s\,dW_s, \quad t \in [0,T]. \tag{7.1}$$

The pair (y,z) is called g-solution with terminal value X *and satisfies in particular $y_t \in \mathbf{L}^2(\mathcal{F}_t)$ for each t. If the driver g satisfies in addition property $(\mathcal{C})$, then $\mathcal{E}^g[X] := y_0$ is called* g-expectation of X *and for each $t \in [0,T]$ there exists a $\mathbb{P}$-a.s. unique $\eta_t \in \mathbf{L}^2(\mathcal{F}_t)$ such that*

$$\mathcal{E}^g[\mathbf{1}_A X] = \mathcal{E}^g[\mathbf{1}_A \eta_t] \quad \text{for all } A \in \mathcal{F}_t.$$

Then $\eta_t = y_t$ and we call $\mathcal{E}^g_t[X] := y_t$ the conditional g-expectation of X under $\mathcal{F}_t$.

Remark 7.4 We can and do choose the process y in Theorem 7.3 continuous, since this will allow us to draw conclusions about the behavior of y_t which hold almost surely, simultaneously for all $t \in [0,T]$, instead of only almost surely almost everywhere. ◊

Next we recall some well-known properties of g-solutions from which we shall deduce conditions on the driver g under which a g-solution describes a time-consistent DMCUF.

Proposition 7.5 *Let g satisfy conditions $(\mathcal{A})$, $(\mathcal{B})$ of Definition 7.2 and denote for any $X \in \mathbf{L}^2$ by (y^X, z^X) the corresponding g-solution as defined in (7.1). Then the following assertions hold:*

a) *$\mathcal{F}_t$-translation invariance: If g satisfies property $(\mathcal{D})$, then*

$$y_t^{X+a_t} = y_t^X + a_t \quad \textit{for any } t \in [0,T] \textit{ and } a_t \in \mathbf{L}^2(\mathcal{F}_t).$$

b) *Monotonicity: For any $X' \in \mathbf{L}^2$ such that $X' \geq X$, we have*

$$y_t^{X'} \geq y_t^X \quad \textit{for any } t \in [0,T].$$

c) *Concavity: If g satisfies property $(\mathcal{E})$, then we have for any $X_1, X_2 \in \mathbf{L}^2$ and any $\beta \in [0,1]$ that*

$$y_t^{\beta X_1 + (1-\beta)X_2} \geq \beta y_t^{X_1} + (1-\beta) y_t^{X_2}.$$

d) *$\mathcal{F}_t$-regularity: For any $X_1, X_2 \in \mathbf{L}^2$ and $A \in \mathcal{F}_t$,*

$$y_t^{\mathbf{1}_A X_1 + \mathbf{1}_{A^c} X_2} = \mathbf{1}_A\, y_t^{X_1} + \mathbf{1}_{A^c}\, y_t^{X_2}.$$

e) *Normalization: If g satisfies property $(\mathcal{C})$, then*

$$y^0 \equiv 0.$$

f) *Positive homogeneity: If g satisfies property $(\mathcal{F})$, then*

$$y_t^{\lambda X} = \lambda y_t^X \quad \textit{for any } \lambda \geq 0.$$

g) *Time-consistency: Let $0 \leq s \leq t \leq T$ and $X_1, X_2 \in \mathbf{L}^2$. Then*

$$y_t^{X_1} = y_t^{X_2} \quad \textit{implies that also} \quad y_s^{X_1} = y_s^{X_2}.$$

Proof For some parts of Proposition 7.5, proofs are available only for the special case that g satisfies in addition to $(\mathcal{A})$ and $(\mathcal{B})$ also

$(\mathcal{C}')$ $g_t(0,0) \equiv 0$.

Therefore we first show how the general case can be reduced to this situation. More precisely, we prove that $(\tilde{y}^X, \tilde{z}^X) := (y^X - y^0, z^X - z^0)$ is the g-solution for the driver

$$\tilde{g}_t(y,z) := g_t(y + y_t^0, z + z_t^0) - g_t(y_t^0, z_t^0), \quad t \in [0,T]$$

and terminal value X. In fact, it is easy to see that $\tilde{g}$ satisfies $(\mathcal{A})$, $(\mathcal{B})$ and $(\mathcal{C}')$. Hence by uniqueness, $(\tilde{y}^X, \tilde{z}^X)$ solves

$$-d\tilde{y}_t^X = \Big(g_t(y_t^X, z_t^X) - g_t(y_t^0, z_t^0) \Big)\, dt - (z_t^X - z_t^0)^*\, dW_t \tag{7.2}$$

$$= \Big(g_t(\tilde{y}_t^X + y_t^0, \tilde{z}_t^X + z_t^0) - g_t(y_t^0, z_t^0) \Big)\, dt - (\tilde{z}_t^X)^*\, dW_t \tag{7.3}$$

$$= \tilde{g}_t(\tilde{y}_t^X, \tilde{z}_t^X)\, dt - (\tilde{z}_t^X)^*\, dW_t. \tag{7.4}$$

Since $\tilde{y}_t^X = y_t^X - y_t^0$ for all $X \in \mathbf{L}^\infty$ and because the properties a) and d) are invariant under the translation by $-y_t^0$, we can thus assume for their proof that g satisfies $(\mathcal{C}')$ as well.

After this preliminary step, the rest is easy:

a) For g satisfying $(\mathcal{C}')$, this can be found in Lemma 4.2 in [BCHMP00]; see also Example 11 in [Pen97].
b) See Proposition 3.5 in [EKPQ97].
c) See Proposition 3.5 in [EKPQ97].
d) If g satisfies $(\mathcal{C}')$, then

$$\mathbf{1}_A\, g_u(\,.\,,\,.\,) = g_u(\mathbf{1}_A\,.\,,\mathbf{1}_A\,.\,) \quad \text{for all } u \geq t \text{ and } A \in \mathcal{F}_t.$$

Hence the claim follows from 2) of the proof of Proposition 36.4 in [Pen97].
e) See Lemma 36.6 in [Pen97].
f) This is Proposition 9 in [RG04]; see also Example 10 in [Pen97].
g) This follows immediately from Proposition 2.5 in [EKPQ97] and the uniqueness of g-solutions.

□

Remark 7.6 To obtain normalization in e) it suffices to have $(\mathcal{C}')$ together with $(\mathcal{A})$ and $(\mathcal{B})$. However, the stronger condition $(\mathcal{C})$ yields in addition that the g-solution is independent of the time horizon. In fact, let us write $\mathcal{E}^g_{t,T}[X]$ instead of $\mathcal{E}^g_t[X]$ to emphasize the dependence on the time horizon T. Then property $(\mathcal{C})$ implies that

$$\mathcal{E}^g_{s,t}[X] = \mathcal{E}^g_{s,T}[X] \quad \text{for } s \leq t \leq T \text{ and } X \in \mathbf{L}^2(\mathcal{F}_t),$$

as described. Note also that $(\mathcal{D})$ implies the equivalence of $(\mathcal{C})$ and $(\mathcal{C}')$. ◊

Since BSDEs are typically defined on $\mathbf{L}^2$ spaces, it appears more natural in this context to define (dynamic) MCUFs on $\mathbf{L}^2$ instead of $\mathbf{L}^\infty$. Thus an MCUF at time t is a mapping from $\mathbf{L}^2$ into $\mathbf{L}^2(\mathcal{F}_t)$ which has all the properties of Definition 3.1 with $\mathbf{L}^\infty$ replaced by $\mathbf{L}^2$ everywhere and with its acceptance set defined as a subset of $\mathbf{L}^2$. In the same way, we extend Definition 3.23 of time-consistency by replacing $\mathbf{L}^\infty$ with $\mathbf{L}^2$. It is easy to check that Lemma 3.25 remains true for DMCUFs on $\mathbf{L}^2$ so that we can make use here of its equivalent conditions for time-consistency. In particular, the following result follows immediately from Proposition 7.5:

Corollary 7.7 *Let g satisfy properties $(\mathcal{A})$, $(\mathcal{B})$, $(\mathcal{D})$ and $(\mathcal{E})$ of Definition 7.2 and denote by (y^X, z^X) the corresponding g-solution with terminal value $X \in \mathbf{L}^2$. Then*

$$\Phi_t(X) := y^X_t, \quad t \in [0,T]$$

defines a time-consistent DMCUF. It is normalized and therefore even strongly time-consistent if g satisfies in addition property $(\mathcal{C})$, and coherent if g also satisfies property $(\mathcal{F})$.

Remark 7.8 i) A similar result, stated for dynamic risk measures, is given in Proposition 25 of [RG04]. However, her definition of a dynamic risk measure (and hence of a DMCUF) differs from ours. For the convenience of the reader, we therefore showed here how Corollary 7.7 can be obtained. We also remark that in her section 4.1.2, Rosazza Gianin states in addition conditions under which the converse holds true, i.e., for when a time-consistent DMCUF can be described by some g-solution.

ii) Note that DMCUFs described by g-solutions are in particular continuous in t.

$\Diamond$

Now we consider an investor whose preferences can be expressed by a DMCUF Φ which is described by a g-solution, and we assume that this investor can trade in some financial market. As in section 6, we want to obtain results for the utility indifference valuation DMCUF by convoluting Φ with the market DMCUF corresponding to the superhedging price process in the given market. To this end, we should like to express also the market DMCUF in terms of BSDEs. However, the superhedging price process (and hence the market DMCUF) is in general not a g-solution, but belongs to the bigger class of (constrained) g-supersolutions which we define next:

Definition 7.9 Let $X \in \mathbf{L}^2$ and let both $\psi : \Omega \times [0,T] \times I\!R \times I\!R^d \to I\!R_+$ and g satisfy $(\mathcal{A})$ and $(\mathcal{B})$ of Definition 7.2. We call a triple (y, z, A) *g-supersolution with terminal value* X if (y, z) is in $\mathbf{M}^2_{\mathbb{F}}(0,T; I\!R) \times \mathbf{M}^2_{\mathbb{F}}(0,T; I\!R^d)$ with y RCLL and $A = (A_t)_{0 \le t \le T}$ is an increasing $\mathbb{F}$-adapted RCLL process with $A_0 = 0$ and $\mathbb{E}\left[A_T^2\right] < \infty$ such that (y, z, A) satisfies

$$y_t \;=\; X + \int_t^T g_s(y_s, z_s)\, ds + (A_T - A_t) - \int_t^T z_s^* \, dW_s, \quad t \in [0,T]. \qquad (7.5)$$

We call the triple a *ψ-constrained g-supersolution* if (y, z, A) satisfies in addition

$$\psi_t(y_t, z_t) = 0 \quad d\mathbb{P} \otimes dt\text{−a.s.}$$

If y satisfies

$$y_t \le y'_t \quad \text{for all } t \in [0,T] \ \mathbb{P}\text{−a.s.}$$

for any ψ-constrained g-supersolution (y', z', A') with terminal value X, we call (y, z, A) the *smallest ψ-constrained g-supersolution with terminal value X*.

Remark 7.10 i) Proposition 1.6 in [Pen99] implies uniqueness of the processes z and A in a g-supersolution (y, z, A) in the following sense: If (y, z', A') is also a g-supersolution with the same terminal value $X \in \mathbf{L}^2$, then z and z' respectively A and A' coincide.

ii) The original terminology in [Pen99] for a ψ-constrained g-supersolution is *g-supersolution under the constraint ψ*. We slightly change the terminology here in order to avoid confusion. In fact, we shall consider the utility indifference valuation for the case of *unconstrained* trading opportunities in the market. But the corresponding market DMCUF will be described by a *ψ-constrained* g-supersolution. The deeper reason for this mismatch is that the construction in terms of ψ-constrained g-supersolutions is somewhat artificial, as we describe the market DMCUF as a stochastic integral with respect to a Brownian motion W and as a process adapted to the filtration generated by W. It would be more natural to use stochastic integrals with respect to the price process S of the traded assets and work with the filtration generated by S.

$\Diamond$

A fundamental result for BSDEs which we require later is the *comparison theorem*. The version we present in Theorem 7.11 can be found in [Pen99], Theorem 1.3:

Theorem 7.11 *Let* $y, y', B, B', g' \in \mathbf{M}^2_{\mathbb{F}}(0,T;I\!R)$ *where* B *and* B' *are RCLL processes with* $B_0 = B'_0$, $\mathbb{E}\left[\sup_{0\le t\le T}|B_t|\right] < \infty$ *and* $\mathbb{E}\left[\sup_{0\le t\le T}|B'_t|\right] < \infty$. *Moreover, let* $z, z' \in \mathbf{M}^2_{\mathbb{F}}(0,T;I\!R^d)$, $X, X' \in \mathbf{L}^2$ *and let* g *be a driver which satisfies* ($\mathcal{A}$) *and* ($\mathcal{B}$). *Assume that* (y, z, B) *solves*

$$y_t = X + \int_t^T g_t(y_t, z_t)\,dt + (B_T - B_t) - \int_t^T z_t^*\,dW_t, \quad t \in [0,T]$$

and (y', z', B') *solves*

$$y'_t = X' + \int_t^T g'_t\,dt + (B'_T - B'_t) - \int_t^T (z'_t)^*\,dW_t, \quad t \in [0,T].$$

If

$$X \ge X', \quad g_t(y'_t, z'_t) \ge g'_t \;\; d\mathbb{P}\otimes dt-\text{a.s.} \quad \text{and} \quad B \succeq B' \text{ (i.e., } B - B' \text{ is increasing)},$$

then we have $\mathbb{P}$*-a.s.*

$$y_t \ge y'_t \quad \text{for all } t \in [0,T].$$

If in addition $\mathbb{P}[X > X'] > 0$ *then* $\mathbb{P}\big[y_t > y'_t \text{ for all } t \in [0,T]\big] > 0$.

In order to define the market functional, we now need to specify the financial market and the set of strategies we allow for trading in the present $\mathbf{L}^2$-setting. We retain the assumptions made at the beginning of this section with respect to the filtered probability space and the d-dimensional Brownian motion W. Our model consists of $n \le d$ risky assets and one riskless asset which is constantly 1 so that the price processes of the n risky assets are already discounted. They are defined by

$$S_t^i \;=\; s_0^i \exp\left(\sum_{j=1}^d \int_0^t \sigma_s^{i,j}\,dW_s^j + \int_0^t \mu_s^i\,ds - \sum_{j=1}^d \frac{1}{2}\int_0^t \left|\sigma_s^{i,j}\right|^2 ds\right), \quad i = 1,\ldots,n$$

with $s_0^i > 0$, $i = 1,\ldots,n$, where μ and σ are uniformly bounded progressively measurable processes and such that the inverse of $\sigma\sigma^*$ exists and is uniformly bounded. Note that there exists an equivalent martingale measure for S so that there are no arbitrage opportunities in this market.

Definition 7.12 An *admissible portfolio* is a triple (x, π, K), where $x \in I\!R$, π is a progressively measurable $I\!R^n$-valued process and K is an adapted RCLL increasing process satisfying $K_0 = 0$ and

$$\mathbb{E}\left[\int_0^T \|\pi_t^*\sigma_t\|^2\,dt + K_T^2\right] < \infty.$$

Here, x is the initial wealth, π_t^i is the amount of money invested in the i-th stock at time t, and K_t is the cumulative consumption up to time t. The corresponding *value process* is defined as the RCLL process $V = (V_t)_{0\le t\le T}$ given by

$$\begin{aligned} dV_t &= \pi_t^*\mu_t\,dt - dK_t + \pi_t^*\sigma_t\,dW_t, \qquad (7.6)\\ V_0 &= x. \end{aligned}$$

An admissible portfolio (x, π, K) is a *hedging portfolio for* $X \in \mathbf{L}^2$ if $V_T = X$, and it is a *minimal hedging portfolio for* X if its value process V satisfies

$$V_t \leq V_t' \quad \text{for all } t \in [0,T] \ \mathbb{P}\text{-a.s.}$$

for every hedging portfolio (x', π', K') for X with value process V'.

Remark 7.13 The definition of an admissible portfolio is here slightly different from the one given in section 5. First of all, due to a different setting, we impose different integrability conditions. Moreover, the process H in an admissible portfolio (x, H, K) in section 5 describes the portfolio by fixing the *numbers* of units of each asset held, whereas in this section the process π fixes the *amounts* invested in each of the assets. The relation between π and H is thus given by $\pi_t^i = H_t^i S_t^i$. ◊

For simplicity we only consider the case of unconstrained hedging in the sense of sections 5 and 6. Thus the agent can use any admissible portfolio for hedging, and the set of payoffs she can superhedge by trading during $(t, T]$ is given by

$$\mathcal{C}_t := \left\{ \int_t^T \pi_u^* \left(\mu_u \, du + \sigma_u \, dW_u\right) - Y \,\middle|\, Y \in \mathbf{L}_+^2 \ , \ (0, \pi, 0) \text{ an admissible portfolio} \right\}. \tag{7.7}$$

Remark 7.14 i) In section 6, we have denoted a typical element of $\mathcal{C}_t$ by g. To avoid confusion with the driver g of a BSDE, we now denote a typical element of $\mathcal{C}_t$ by h.

ii) In principle, the present approach via the results on minimal g-supersolutions can be extended to more general situations with constraints imposed on trading, i.e., when the set $\mathcal{C}_t'$ of payoffs which can be superhedged by trading during $(t, T]$ is a subset of the above $\mathcal{C}_t$. This idea goes back to Bender and Kohlmann [BK04] who also give many examples of general constraints. For the applications here, we need $\mathcal{C}_t'$ to be convex so that we can impose only convex constraints.

◊

Similar to section 6, the programme for describing the utility indifference valuation p with respect to Φ and the market corresponding to the family $(\mathcal{C}_t)$ now looks as follows:

1) Construct the market DMCUF corresponding to $(\mathcal{C}_t)$; compare (6.8).
2) Describe it via BSDEs.
3) Convolute it with Φ to obtain U^{opt}; compare (6.3) and (6.9).
4) Describe U^{opt} via BSDEs.
5) Express p via U^{opt}; compare (6.4).

Because we work here in $\mathbf{L}^2$ instead of $\mathbf{L}^\infty$, the above steps become technically slightly different. The main problem is that we cannot construct the market DMCUF on all of $\mathbf{L}^2$. But fortunately, the convolution with Φ can still be formed since it only needs the values of the market DMCUF on a suitable subset of $\mathbf{L}^2$; this essentially goes back to the last equality of (4.4) in Theorem 4.3. Let us explain this in more detail.

In analogy to (6.8), we should want to define the market DMCUF by

$$\Phi_t^{-\mathcal{C}_t}(X) := \operatorname{ess\,sup} \left\{ m_t \in \mathbf{L}^2(\mathcal{F}_t) \,\middle|\, X - m_t \in -\mathcal{C}_t \right\} \tag{7.8}$$

so that a simple reformulation would give

$$-\Phi_t^{-\mathcal{C}_t}(-X) = \operatorname{ess\,inf}\left\{ m_t \in \mathbf{L}^2(\mathcal{F}_t) \,\middle|\, X = m_t + g \text{ for some } g \in \mathcal{C}_t \right\}. \tag{7.9}$$

In other words, $-\Phi_t^{-\mathcal{C}_t}(-X)$ should correspond to the superhedging price of X at time t. But this does not work with every X in $\mathbf{L}^2$. It is well-known that in contrast to the $\mathbf{L}^\infty$-context, a hedging portfolio need not exist for every $X \in \mathbf{L}^2$ in general so that the set on the RHS of (7.9) can be empty and the essential infimum is possibly not well-defined. In particular, $\Phi_t^{-\mathcal{C}_t}$ in (7.8) is not an MCUF at time t because it is not defined on all of $\mathbf{L}^2$. We might try to save the situation by defining the essential infimum in (7.9) as ∞ when it is taken over an empty set. But in view of the desired interpretation as superhedging price, this is not appropriate either. In fact, there might exist a set $A \in \mathcal{F}_t$ with $\mathbb{P}[A] > 0$ and such that there exists a hedging portfolio for X on A, i.e., for $X\mathbf{1}_A$, and then the superhedging price of X at time t should be finite on A. Hence the definition (7.8) cannot be used for every $X \in \mathbf{L}^2$; we must restrict X to some suitable subset of $\mathbf{L}^2$.

Now the reason why we consider the functional $\Phi_t^{-\mathcal{C}_t}$ is that we want to convolute it with the DMCUF Φ which expresses the agent's preferences. Fortunately, this operation does not need the values of $\Phi_t^{-\mathcal{C}_t}$ on all of $\mathbf{L}^2$; this can be seen from (4.4) which is easily extended from the $\mathbf{L}^\infty$- to the present $\mathbf{L}^2$-context. In more detail, (4.4) suggests that we should have

$$U_t^{\text{opt}}(X) = \text{“}\Phi_t \Box \Phi_t^{-\mathcal{C}_t}(X)\text{”} = \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \Big(\Phi_t(X+Y) + \Phi_t^{-\mathcal{C}_t}(-Y) \Big) \tag{7.10}$$

for all $X \in \mathbf{L}^2$, where $\mathcal{B}$ is an arbitrary subset of $\mathbf{L}^2$ such that

$$\mathcal{B} \supseteq \left\{ Y \in \mathbf{L}^2 \,\middle|\, \Phi_t^{-\mathcal{C}_t}(Y) \text{ from (7.8) is well-defind in } \mathbf{L}^2 \text{ and } \geq 0 \right\}.$$

In other words, $\mathcal{B}$ should contain the “acceptance set of $\Phi_t^{-\mathcal{C}_t}$”. To prove that (7.10) is indeed true with $\mathcal{B} := -\mathcal{C}_0$, we shall first show that the superhedging price at time t for $X \in \mathcal{C}_0$ coincides with the RHS of (7.9), and is ≤ 0 if $X \in \mathcal{C}_t \subseteq \mathcal{C}_0$; hence $\Phi_t^{-\mathcal{C}_t}(X)$ is well-defined by (7.8) for $X \in -\mathcal{C}_0$ and ≥ 0 if $X \in -\mathcal{C}_t$. Then we prove that

$$U_t^{\text{opt}}(X) := \operatorname*{ess\,sup}_{h \in \mathcal{C}_t} \Phi_t(X+h) \tag{7.11}$$

coincides for every $X \in \mathbf{L}^2$ with the RHS of (7.10) for $\mathcal{B} := -\mathcal{C}_0$.

The next result achieves steps 1) and 2) in the above scheme. It shows that the superhedging price process for $X \in \mathbf{L}^2$ can be described via a constrained g-supersolution of a BSDE, and that this process is nonpositive at t if and only if $X \in \mathcal{C}_t$. Moreover, the superhedging price operator is shown to coincide with $-\Phi_t^{-\mathcal{C}_t}(-\,.\,)$ from (7.9) or (7.8) on $\mathcal{C}_0$. Note again that in contrast to the $\mathbf{L}^\infty$ case, these results do not hold on all of $\mathcal{C}_t + \mathbf{L}^2(\mathcal{F}_t) \supseteq \mathcal{C}_0$, because not every $X \in \mathbf{L}^2(\mathcal{F}_t)$ admits a superhedging portfolio.

Proposition 7.15 a) *Let $X \in \mathbf{L}^2$ be such that there exists a hedging portfolio for X. Then the minimal hedging portfolio $(\tilde{x}, \tilde{\pi}, \tilde{A})$ for X exists, and the corresponding value process $\tilde{V}$ coincides with the y-component from the smallest ψ-constrained g-supersolution of the BSDE*

$$-dy_t = g_t^m(z_t)\,dt + dA_t - z_t^*\,dW_t \tag{7.12}$$

with terminal value

$$y_T = X$$

and constraint

$$\psi_t(z_t) := \left\| z_t - \sigma_t^* \left(\sigma_t \sigma_t^*\right)^{-1} \sigma_t z_t \right\| = 0 \quad d\mathbb{P} \otimes dt\text{-a.s.}, \tag{7.13}$$

where

$$g_t^m(z) := -z^* \sigma_t^* \left(\sigma_t \sigma_t^*\right)^{-1} \mu_t. \tag{7.14}$$

b) *For $X \in \mathbf{L}^2$, the minimal hedging portfolio exists and has a value process $\tilde{V}$ which satisfies $\tilde{V}_t \leq 0$ if and only if X belongs to $\mathcal{C}_t$.*

c) *For any $h^0 \in \mathcal{C}_0 \supseteq \mathcal{C}_t$, the minimal hedging portfolio $(\tilde{x}, \tilde{\pi}, \tilde{K})$ exists, and its value process $\tilde{V}$ coincides with $(-\Phi_t^{-\mathcal{C}_t}(-h_0))_{0 \leq t \leq T}$ with $-\Phi_t^{-\mathcal{C}_t}(-\,.\,)$ from (7.9): For each $t \in [0, T]$, we have*

$$\tilde{V}_t = -\Phi_t^{-\mathcal{C}_t}(-h^0). \tag{7.15}$$

Remark 7.16 i) We denote the driver in (7.12) by $g^m(\,.\,)$ to emphasize its connection to the market functional (respectively to $-\Phi_t^{-\mathcal{C}_t}(-\,.\,)$).

ii) Since $\sigma_t^* \left(\sigma_t \sigma_t^*\right)^{-1} \sigma_t$ is the projection onto the range of σ_t^*, the constraint (7.13) simply ensures that z_t is in the range of σ_t^*. As mentioned before, this is needed because our strategies ought to be expressed via S, not W.

◊

Proof a) We first show that (x, π, K) is a hedging portfolio for X with value process V if and only if $(V, \sigma^* \pi, K)$ is a ψ-constrained g-supersolution with terminal value X. To see this, note that (7.5) can equivalently be written as

$$-dy_t = g_t(y_t, z_t)\, dt + dA_t - z_t^*\, dW_t, \quad y_T = X$$

and that the constraint from ψ is always satisfied for $z = \sigma^* \pi$. Hence we only have to check the integrability conditions in Definitions 7.9 and 7.12, and show that for any ψ-constrained g-supersolution (y, z, A) we can write $z = \sigma^* \pi$ for a suitable process π. However, the latter holds since z satisfies the constraint from ψ so that we can take $\pi := (\sigma\sigma^*)^{-1} \sigma z$, and the former holds since μ, σ and $(\sigma\sigma^*)^{-1}$ are uniformly bounded processes. Now the assertion follows if we can prove the existence of a smallest ψ-constrained g-supersolution with terminal condition X. But this follows by Theorem 4.2 of [Pen99] already from the existence of a ψ-constrained g-supersolution with terminal value X or, equivalently, from the existence of a hedging portfolio for X.

b) Any $X \in \mathcal{C}_t$ is of the form $X = \int_t^T \pi_u^* \left(\mu_u\, du + \sigma_u\, dW_u\right) - Y$ where $Y \in \mathbf{L}_+^2$ and π is a progressively measurable $I\!R^d$-valued process such that

$$\mathbb{E}\left[\int_0^T \|\pi_t^* \sigma_t\|^2\, dt\right] < \infty.$$

Hence $(0, \pi', K')$ with $K'_u := 0$ for $u < T$, $K'_T := Y$, $\pi' = 0$ on $[\![0, t]\!]$ and $\pi' = \pi$ on $]\!]t, T]\!]$ is a hedging portfolio for X so that by a) the minimal hedging portfolio for X exists. Moreover, since the value process V' of $(0, \pi', K')$ satisfies $V'_t = 0$, the value process $\tilde{V}$ of the minimal hedging portfolio for X satisfies $\tilde{V}_t \leq 0$. To finish the proof of b), it suffices to show that if for $X \in \mathbf{L}^2$ the minimal hedging portfolio

(x, π, K) exists with value process V such that $V_t \le 0$, then $X \in \mathcal{C}_t$. But this is easy since (7.6) implies that

$$X = V_T = (V_T - V_t) + V_t = \int_t^T \pi_u^* \left(\mu_u \, du + \sigma_u \, dW_u\right) - (K_T - K_t - V_t)$$

where $K_T - K_t - V_t \in \mathbf{L}^2_+$ so that $X \in \mathcal{C}_t$.

c) By part b), it suffices to prove (7.15). Also by b) the minimal hedging portfolio $(\tilde{x}, \tilde{\pi}, \tilde{K})$ for h^0 exists. If $\tilde{V}$ denotes the corresponding value process, we can write

$$h^0 = \tilde{V}_t + \int_t^T \tilde{\pi}_u^* (\mu_u \, du + \sigma_u \, dW_u) - (\tilde{K}_T - \tilde{K}_t) =: \tilde{V}_t + \tilde{h},$$

where $\tilde{h} \in \mathcal{C}_t$. This yields the estimate

$$\begin{aligned} -\Phi_t^{-\mathcal{C}_t}(-h^0) &= \operatorname{ess\,inf} \left\{ m_t \in \mathbf{L}^2(\mathcal{F}_t) \middle| \, h^0 = m_t + h' \quad \text{for some } h' \in \mathcal{C}_t \right\} \qquad (7.16) \\ &\le \tilde{V}_t. \end{aligned}$$

The converse inequality is shown by contradiction. Suppose it does not hold. Then there exist $\epsilon > 0$ and $A \in \mathcal{F}_t$ with $\mathbb{P}[A] > 0$ such that

$$-\Phi_t^{-\mathcal{C}_t}(-h^0) + \epsilon < \tilde{V}_t \quad \text{on } A.$$

By (7.16), we can then find $\overline{m}_t \in \mathbf{L}^2(\mathcal{F}_t)$ and $\overline{h} \in \mathcal{C}_t$ such that $h^0 = \overline{m}_t + \overline{h}$ and some set $B \subseteq A$ with $B \in \mathcal{F}_t$ and $\mathbb{P}[B] > 0$ so that

$$\overline{m}_t \le -\Phi_t^{-\mathcal{C}_t}(-h^0) + \epsilon < \tilde{V}_t = \tilde{V}_{t-} + \Delta\tilde{V}_t = \tilde{V}_{t-} - \Delta\tilde{K}_t \le \tilde{V}_{t-} \quad \text{on } B. \qquad (7.17)$$

By b) the minimal hedging portfolio $(\overline{x}, \overline{\pi}, \overline{K})$ for $\overline{h} \in \mathcal{C}_t$ exists and if we denote the corresponding value process by $\overline{V}$, we can write

$$\overline{h} = \overline{V}_t + \int_t^T \overline{\pi}_s^* (\mu_s \, ds + \sigma_s \, dW_s) - \overline{K}_T + \overline{K}_t. \qquad (7.18)$$

Now we fix $t \in (0, T)$ and construct a new hedging portfolio $(\hat{x}, \hat{\pi}, \hat{K})$ for h^0 such that its value process $\hat{V}$ satisfies $\hat{V}_t = \overline{m}_t < \tilde{V}_t$ on B which contradicts the minimality of $\tilde{V}$. To this end we define

$$\begin{aligned} \hat{x} &:= \tilde{x}, \\ \hat{\pi} &:= \tilde{\pi} \mathbf{1}_{[\![0,t]\!]} + (\overline{\pi} \mathbf{1}_B + \tilde{\pi} \mathbf{1}_{B^c}) \mathbf{1}_{]\!]t,T]\!]}, \\ \hat{K} &:= \tilde{K} \quad \text{on } [\![0, t[\![, \\ \hat{K}_u &:= \left(\tilde{V}_{t-} + \tilde{K}_{t-} - \overline{m}_t + \overline{K}_u - \overline{K}_t - \overline{V}_t \mathbf{1}_{\{u=T\}} \right) \mathbf{1}_B + \tilde{K}_u \mathbf{1}_{B^c} \\ &\quad\ \text{for } t \le u \le T. \end{aligned}$$

We note that $\tilde{V}_{t-} + \tilde{K}_{t-} = \tilde{x} + \int_0^t \tilde{\pi}_s^* (\mu_s \, ds + \sigma_s \, dW_s)$ so that

$$\hat{V}_t = \hat{x} + \int_0^t \hat{\pi}_s^* \left(\mu_s \, ds + \sigma_s \, dW_s\right) - \hat{K}_t = \overline{m}_t \quad \text{on } B \qquad (7.19)$$

and that by (7.18), we have on B from the definition of $\hat{K}_T$ that

$$\begin{aligned}\hat{V}_T &= \tilde{V}_{t-} + \int_t^T \overline{\pi}_s^*(\mu_s\,ds + \sigma_s\,dW_s) - \hat{K}_T + \tilde{K}_{t-} \\ &= \overline{m}_t + \overline{h} \\ &= h^0.\end{aligned}$$

Hence we are only left to show that $(\hat{x}, \hat{\pi}, \hat{K})$ is an admissible portfolio, which is obviously true if $\hat{K}$ is increasing. Because $\overline{h} \in \mathcal{C}_t$ implies that $\overline{V}_t \leq 0$, this is obvious if $\Delta\hat{K}_t := \hat{K}_t - \hat{K}_{t-} = \hat{K}_t - \tilde{K}_{t-} \geq 0$ on B. However, the latter holds true since by (7.19) and (7.17) we have $\Delta\hat{K}_t = -\Delta\hat{V}_t = -(\overline{m}_t - \tilde{V}_{t-}) > 0$ on B. This establishes the contradiction and hence completes the proof.

□

We now pass on to steps 3) – 5) in our scheme. So let us fix a DMCUF Φ and define for each $X \in \mathbf{L}^2$ the utility indifference value $p_t(X)$ at time t implicitly by

$$\operatorname*{ess\,sup}_{h \in \mathcal{C}_t} \Phi_t(x_t + h) = \operatorname*{ess\,sup}_{h \in \mathcal{C}_t} \Phi_t(x_t - p_t(X) + X + h), \tag{7.20}$$

where $x_t \in \mathbf{L}^2(\mathcal{F}_t)$ is the initial endowment at time t. We also define $U_t^{\mathrm{opt}}(X) = \operatorname*{ess\,sup}_{h \in \mathcal{C}_t} \Phi_t(X + h)$ as in (7.11). If

$$U_t^{\mathrm{opt}}(0) \in \mathbf{L}^2(\mathcal{F}_t) \quad \text{and} \quad U_t^{\mathrm{opt}}(X) \in \mathbf{L}^2(\mathcal{F}_t),$$

then we can use the translation invariance of Φ_t to solve (7.20) for $p_t(X)$ and get

$$p_t(X) = U_t^{\mathrm{opt}}(X) - U_t^{\mathrm{opt}}(0) \in \mathbf{L}^2(\mathcal{F}_t). \tag{7.21}$$

This last expression is a first answer to step 5). For steps 3) and 4), we assume that Φ is described by some g-solution and we should also like to express $U^{\mathrm{opt}}(X)$ in terms of BSDEs. The idea to achieve this is as follows. Thanks to Proposition 7.15 and (7.10), we know that U_t^{opt} is "morally" the convolution of Φ_t with the market MCUF $\Phi_t^{-\mathcal{C}_t}$. Now Barrieu and El Karoui have proved in [BEK04] that the convolution of DMCUFs which are both described by g-solutions corresponds (under some technical assumptions) to the g-solution whose driver is the pointwise convolution (in the sense of Rockafellar as in (4.3)) at each time t of the drivers for the two original g-solutions. Since the market functional is not a g-solution but a constrained g-supersolution, we have here a slightly different setting. Nevertheless, we can extend the result of Barrieu and El Karoui to this more general setting by similar arguments.

Theorem 7.17 a) *Let the DMCUF Φ be described by a g-solution with a driver g which satisfies ($\mathcal{A}$), ($\mathcal{B}$), ($\mathcal{D}$) and ($\mathcal{E}$). With g^m as in (7.14), define for $z \in I\!R^d$*

$$\hat{g}_t(z) := \sup_{v \in I\!R^n} \{g_t(z + \sigma_t^* v) + g_t^m(-\sigma_t^* v)\} = \sup_{v \in I\!R^n} \{g_t(z + \sigma_t^* v) + v^* \mu_t\} \tag{7.22}$$

and fix $X \in \mathbf{L}^2$. If $\hat{g} : \Omega \times [0,T] \times I\!R^d \to I\!R$ satisfies ($\mathcal{A}$) and ($\mathcal{B}$), then the g-solution $(\hat{y}, \hat{z})$ of

$$-d\hat{y}_t = \hat{g}_t(\hat{z}_t)\,dt - \hat{z}_t^*\,dW_t, \quad \hat{y}_T = X$$

exists. If in addition there exists $\overline{z} \in \mathbf{M}^2_{\mathbb{F}}(0,T;I\!R^d)$ *satisfying the* ψ*-constraint (7.13) and such that*

$$\hat{g}_t(\hat{z}_t) = g_t(\hat{z}_t + \overline{z}_t) + g^m_t(-\overline{z}_t) \quad d\mathbb{P} \otimes dt\text{−a.s.}, \tag{7.23}$$

then

$$U^{\mathrm{opt}}_t(X) = \operatorname*{ess\,sup}_{h^0 \in \mathcal{C}_0} \left\{ \Phi_t(X + h^0) + \Phi^{-\mathcal{C}_t}_t(-h^0) \right\} = \hat{y}_t. \tag{7.24}$$

In other words, $U^{\mathrm{opt}}(X)$ *then equals the* y*-component of the* g*-solution with driver* $\hat{g}$ *and terminal value* X*.*

b) *Suppose the assumptions of a) hold and denote by* (y^0, z^0) *the* g*-solution with driver* $\hat{g}$ *and terminal value 0. If there exists* $\overline{z}^0 \in \mathbf{M}^2_{\mathbb{F}}(0,T;I\!R^d)$ *satisfying the* ψ*-constraint (7.13) and also*

$$\hat{g}_t(z^0_t) = g_t(z^0_t + \overline{z}^0_t) + g^m_t(-\overline{z}^0_t),$$

then $\big(p_t(X)\big)_{0 \le t \le T}$ *is the* g*-expectation with driver*

$$\tilde{\hat{g}}_t(z) := \hat{g}_t(z + z^0_t) - \hat{g}_t(z^0_t) \tag{7.25}$$

and terminal value X*.*

Remark 7.18 i) It is easy to check that $\hat{g}$ always satisfies $(\mathcal{D})$ and $(\mathcal{E})$; the latter holds since g and g^m are both concave and for any $\beta \in [0,1]$, we can replace the supremum over all $v \in I\!R^n$ in (7.22) by the supremum over all elements $\beta v^1 + (1-\beta)v^2$, where $v^1, v^2 \in I\!R^n$. Moreover $\hat{g}$ can always be chosen product-measurable on $\Omega \times [0,T] \times I\!R^d$ and such that $(\omega,t) \mapsto \hat{g}(\omega,t,z)$ is $\mathbb{F}$-progressively measurable, so that $(\mathcal{A})$ is reduced to an integrability condition. In fact, we can fix a product-measurable $A \subseteq \Omega \times [0,T]$ such that A^c is a $d\mathbb{P} \otimes dt$-nullset and $z \mapsto g(\omega,t,z)$ is continuous on $I\!R^d$ for all $(\omega,t) \in A$. Without loss of generality, $\mathbf{1}_A$ is $\mathbb{F}$-adapted; otherwise replace it by $A' := A \cap (B \times \Omega)$, where $B := \{t \in [0,T] \,|\, \mathbb{E}[\mathbf{1}_A(\omega,t)] = 1\}$ is a Borel set. It follows from $\mathbb{E}\left[\int_0^T \mathbf{1}_A(\omega,t)\,dt\right] = T$ and Fubini's theorem that $\int_0^T \mathbf{1}_B(t)\,dt = T$ $\mathbb{P}$-a.s. so that $(A')^c$ is a $d\mathbb{P} \otimes dt$-nullset. Adaptedness of $\mathbf{1}_{A'}$ is then implied by the usual conditions and since $\mathbb{E}[\mathbf{1}_{A'}(\omega,t)] \in \{0,1\}$ for each $t \in [0,T]$. Now, since $\mathbf{1}_A$ is product-measurable and adapted, it has a progressively measurable modification $Y = (Y_t)_{0 \le t \le T}$. Define

$$\overline{g}_t(z) := Y_t \sup_{v \in \mathbb{Q}^n} \{ g_t(z + \sigma^*_t v) + v^* \mu_t \}$$

on $\Omega \times [0,T] \times I\!R^d$. Then $\overline{g}$ is product-measurable on $\Omega \times [0,T] \times I\!R^d$ and $(\omega,t) \mapsto \overline{g}(\omega,t,z)$ is progressively measurable. Finally, we need to show that

$$\hat{g}_t(z) = \overline{g}_t(z) \quad \text{for all } z \in I\!R^d \quad d\mathbb{P} \otimes dt\text{-a.s.}$$

To this end, note that by Fubini's theorem and since Y is a modification of $\mathbf{1}_A$ where A^c is a $d\mathbb{P} \otimes dt$-nullset we have $Y\mathbf{1}_A = 1$ $d\mathbb{P} \otimes dt$-a.s. Hence we can conclude that $d\mathbb{P} \otimes dt$-a.s.

$$\hat{g}_t(z) = Y\mathbf{1}_A \hat{g}_t(z) = Y\mathbf{1}_A \sup_{v \in \mathbb{Q}^n} \{ g_t(z + \sigma^*_t v) + v^* \mu_t \} = \overline{g}_t(z) \quad \text{for all } z \in I\!R^d,$$

where the second equality holds since g is continuous in z on A.

ii) If $\hat{g}$ satisfies $(\mathcal{A})$ and $(\mathcal{B})$, it is by Corollary 7.7 the driver of a g-solution which describes a time-consistent DMCUF. Note that the condition (7.23) on $\overline{z}$ depends on X via $\hat{z}$. If it does not hold for all $X \in \mathbf{L}^2$, steps 1) and 2) in the following proof still show that $\hat{y} = \hat{y}(X)$ is an upper bound for $U^{\text{opt}}(X)$. However, $\hat{y}(\,.\,)$ need not describe $U^{\text{opt}}(\,.\,)$ on all of $\mathbf{L}^2$ because the upper bound need not be attained.

iii) Suppose $p_{\cdot,T} = \mathcal{E}^{\tilde{\hat{g}}}_{\cdot,T}$ is described by the g-expectation with driver $\tilde{\hat{g}}$ on all of $\mathbf{L}^2$. Since $\tilde{\hat{g}}$ satisfies $(\mathcal{C})$, we know from Remark 7.6 that

$$\mathcal{E}^{\tilde{\hat{g}}}_{s,t}[X] = \mathcal{E}^{\tilde{\hat{g}}}_{s,T}[X] \quad \text{for all } s \le t \le T \text{ and } X \in \mathbf{L}^2(\mathcal{F}_t).$$

Since DMCUFs defined via BSDEs are always time-consistent, one might be tempted to conclude that the family p satisfies the recursiveness property

$$(\mathcal{R}) \;\; p_{s,t}(p_{s,T}(X)) = p_{s,T}(X) \quad \text{for all } s \le t \le T \text{ and } X \in \mathbf{L}^2$$

introduced in section 3. But how is $p_{\cdot,t}$ defined? In view of the desired interpretation, we should take $p_{\cdot,t} = U^{\text{opt}}_{\cdot,t} - U^{\text{opt}}_{\cdot,t}(0)$, where $U^{\text{opt}}_{\cdot,t}$ is described by the g-solution with driver $\hat{g}$ and time horizon t, and then ask if $p_{\cdot,t}$ coincides with $\mathcal{E}^{\tilde{\hat{g}}}_{\cdot,t}$. In general this is not true: Because $\tilde{\hat{g}}$ depends on z^0 which itself depends on the time horizon T, $p_{\cdot,t}$ will in general correspond to a g-expectation with a driver different from $\tilde{\hat{g}}$. However, if the driver $\hat{g}$ corresponding to U^{opt} is deterministic, one can show that

$$U^{\text{opt}}_{s,t}(X) = U^{\text{opt}}_{s,T}(X) - U^{\text{opt}}_{t,T}(0) \quad \text{for all } s \le t \le T \text{ and } X \in \mathbf{L}^2(\mathcal{F}_t).$$

This implies that

$$p_{s,t}(X) = U^{\text{opt}}_{s,t}(X) - U^{\text{opt}}_{s,t}(0) = U^{\text{opt}}_{s,T}(X) - U^{\text{opt}}_{s,T}(0) = p_{s,T}(X)$$

for $s \le t \le T$ and $X \in \mathbf{L}^2(\mathcal{F}_t)$ so that the time-consistency of $p_{\cdot,T}$ does imply $(\mathcal{R})$ after all. Example 7.19 below and the subsequent remark illustrate that p can satisfy $(\mathcal{R})$ even if $\hat{g}$ is not deterministic. It would be nice to have also an explicit example for p described by a g-solution where $(\mathcal{R})$ does not hold.

$\diamond$

Proof of Theorem 7.17

a) 1) We first show the first equality in (7.24), i.e., that

$$\operatorname*{ess\,sup}_{h \in \mathcal{C}_t} \Phi_t(X+h) = \operatorname*{ess\,sup}_{h' \in \mathcal{C}_0} \left\{ \Phi_t(X+h') + \Phi_t^{-\mathcal{C}_t}(-h') \right\}. \tag{7.26}$$

Since $\mathcal{C}_t \subseteq \mathcal{C}_0$ and $\Phi_t^{-\mathcal{C}_t}$ is non-negative on $-\mathcal{C}_t$ by part b) of Proposition 7.15, the inequality "$\le$" is trivial. The converse inequality follows if for any $h' \in \mathcal{C}_0$, we have $\Phi_t^{-\mathcal{C}_t}(-h') \in \mathbf{L}^2(\mathcal{F}_t)$ and $h := h' + \Phi_t^{-\mathcal{C}_t}(-h') \in \mathcal{C}_t$, since then $\mathcal{F}_t$-translation invariance of Φ_t implies that

$$\Phi_t(X+h') + \Phi_t^{-\mathcal{C}_t}(-h') = \Phi_t(X+h).$$

To show that these two properties hold, we recall from Proposition 7.15 that the minimal hedging portfolio (x', π', K') exists for $h' \in \mathcal{C}_0$ and that its value V'_t at time t equals $-\Phi_t^{-\mathcal{C}_t}(-h')$ so that in particular $\Phi_t^{-\mathcal{C}_t}(-h') \in \mathbf{L}^2(\mathcal{F}_t)$. Again by Proposition 7.15, $h \in \mathcal{C}_t$ if and only if a hedging portfolio for h exists with a non-positive value at time t. But since $V'_t = -\Phi_t^{-\mathcal{C}_t}(-h')$ and (x', π', K') is a hedging portfolio for h', we have

$$h' = -\Phi_t^{-\mathcal{C}_t}(-h') + \int_t^T (\pi'_s)^*(\mu_s\, ds + \sigma_s\, dW_s) - (K'_T - K'_t).$$

Hence

$$h = \int_t^T (\pi'_s)^*(\mu_s\, ds + \sigma_s\, dW_s) - (K'_T - K'_t)$$

admits the hedging portfolio $(0, \pi, K)$ with $\pi := \pi' \mathbf{1}_{]\!]t,T]\!]}$ and $K := K' \mathbf{1}_{[\![t,T]\!]}$ which has value 0 at time t. This proves (7.26).

2) To show the second equality in (7.24) we take $h' \in \mathcal{C}_0$ and denote by (y, z) the g-solution for the driver g and terminal value $X + h'$. By Proposition 7.15 the process $(-\Phi_t^{-\mathcal{C}_t}(-h'))_{0 \le t \le T}$ is the y-component of the smallest ψ-constrained g-supersolution (y', z', A') with ψ from (7.13), driver g^m and terminal value h'. Hence we get

$$\begin{aligned} -d\left(\Phi_t(X + h') + \Phi_t^{-\mathcal{C}_t}(-h')\right) &= \left(g_t(z_t) - g_t^m(z'_t)\right) dt - dA'_t - \left(z_t - z'_t\right)^* dW_t \\ &= \left(g_t(\tilde z_t + z'_t) + g_t^m(-z'_t)\right) dt - dA'_t - \tilde z_t^*\, dW_t, \quad (7.27) \\ \Phi_T(X + h') + \Phi_T^{-\mathcal{C}_T}(-h') &= X, \end{aligned}$$

where we set $\tilde z_t := z_t - z'_t$ and use that $-g_t^m(\,.\,) = g_t^m(-\,.\,)$. Since

$$\hat g_t(\tilde z_t) \ge g_t(\tilde z_t + z'_t) + g_t^m(-z'_t) \;\; d\mathbb{P} \otimes dt\text{-a.s. and } 0 \succeq -A',$$

the comparison result in Theorem 7.11 applied to the driver $\hat g_t(\,.\,)$ with solution $(\hat y, \hat z, 0)$ and the integrand $g_t(\tilde z_t + z'_t) + g_t^m(-z'_t)$ with solution $\Big(\big(\Phi_t(X + h') + \Phi_t^{-\mathcal{C}_t}(-h')\big)_t, \tilde z, -A'\Big)$ yields

$$\hat y_t \ge \Phi_t(X + h') + \Phi_t^{-\mathcal{C}_t}(-h') \quad \text{for all } t \in [0, T] \; \mathbb{P}\text{-a.s.} \tag{7.28}$$

Hence $\hat y$ is an upper bound for U^{opt}.

3) Next we construct an element $\check h \in \mathcal{C}_0$ for which this bound is attained to establish the second equality in (7.24). To this end set

$$\check h := \int_0^T g_t^m(-\overline{z}_t)\, dt + \int_0^T \overline{z}_t^*\, dW_t$$

and note that $\overline{z}$ by assumption satisfies the constraint ψ from (7.13). Hence $\pi_t := (\sigma_t \sigma_t^*)^{-1} \sigma_t \overline{z}_t$, $t \in [0, T]$, satisfies $\pi_t^* \sigma_t = \overline{z}_t^*$ so that

$$\check h = \int_0^T \pi_t^*(\mu_t\, dt + \sigma_t\, dW_t).$$

Thus $(0, \pi, 0)$ is a hedging portfolio for $\check h$ and so $\check h \in \mathcal{C}_0$ by Proposition 7.15. Next we define $(\check y_t)_{0 \le t \le T}$ as the continuous process

$$\check y_t := \int_0^t g_s^m(-\overline{z}_s)\, ds + \int_0^t \overline{z}_s^*\, dW_s.$$

Again since $g_t^m(-\,.\,) = -g_t^m(\,.\,)$, $(\check{y}, \overline{z})$ is the unique g-solution of

$$-d\check{y}_t = g_t^m(\overline{z}_t)\,dt - \overline{z}_t^*\,dW_t, \quad \check{y}_T = \check{h}.$$

In particular, since $\overline{z}$ satisfies the constraint ψ from (7.13), the comparison result in Theorem 7.11 implies that the triple $(\check{y}, \overline{z}, 0)$ is the smallest ψ-constrained g-supersolution with terminal value $\check{h}$ and driver g^m. Therefore and since $\check{h} \in \mathcal{C}_0$, parts c) and a) of Proposition 7.15 yield that $-\check{y}_t = \Phi_t^{-\mathcal{C}_t}(-\check{h})$. We already know from (7.27) that for $h' := \check{h}$ we have

$$-d\left(\Phi_t(X + \check{h}) + \Phi_t^{-\mathcal{C}_t}(-\check{h})\right) = \left(g_t(\tilde{z}_t + \overline{z}_t) + g_t^m(-\overline{z}_t)\right)\,dt - \tilde{z}_t^*\,dW_t$$

for some $\tilde{z} \in \mathbf{M}^2_{\mathbb{F}}(0, T; I\!R^d)$. Since one can easily check that the driver $(g_t(\,.\, + \overline{z}_t) + g_t^m(-\overline{z}_t))_{0 \le t \le T}$ satisfies $(\mathcal{A})$ and $(\mathcal{B})$ and since by assumption (7.23) we have

$$\hat{g}_t(\hat{z}_t) = g_t(\hat{z}_t + \overline{z}_t) + g_t^m(-\overline{z}_t) \quad d\mathbb{P} \otimes dt\text{–a.s.},$$

uniqueness of g-solutions implies that

$$\Phi_t(X + \check{h}) + \Phi_t^{-\mathcal{C}_t}(-\check{h}) = \hat{y}_t \quad \text{for all } t \in [0, T]\ \mathbb{P}\text{-a.s.}$$

Hence $\hat{y}$ is not only an upper bound for U^{opt}, but equal to it. This proves (7.24).

b) With a) and (7.21), this follows from the uniqueness of g-solutions and since

$$\begin{aligned} -d\left(y_t^X - y_t^0\right) &= \left(\hat{g}_t\left(z_t^X\right) - \hat{g}_t\left(z_t^0\right)\right)\,dt - \left(z_t^X - z_t^0\right)^*\,dW_t \\ &= \left(\hat{g}_t\left(\tilde{z}_t + z_t^0\right) - \hat{g}_t\left(\left(z_t^0\right)\right)\right)\,dt - \left(\tilde{z}_t\right)^*\,dW_t, \\ y_T^X - y_T^0 &= X, \end{aligned}$$

where $\tilde{z} := z^X - z^0$.

□

We conclude this section with an explicit example where the DMCUF Φ is given by the conditional exponential certainty equivalent with risk aversion γ, i.e.,

$$\Phi_t(X) := -\frac{1}{\gamma}\log \mathbb{E}[\exp(-\gamma X)\,|\mathcal{F}_t] \quad \text{for } X \text{ sufficiently integrable;}$$

see also Example 3.3. Then Φ is described by the g-solution with driver $g_t(z) := -\frac{\gamma}{2}\|z\|^2$; see, e.g., section 3.1 in [BEK04]. Although this driver obviously does not satisfy $(\mathcal{B})$, so that Theorem 7.17 cannot be applied, this is quite an illustrative example. In fact, the driver of the g-solution describing the utility indifference value process is known here explicitly, and we can show by formal calculations that it corresponds to $\tilde{\hat{g}}$ from (7.25). Moreover, this example shows that if g has a nice structure, one can eliminate the dependence of $\tilde{\hat{g}}$ on z^0 by expressing the value process as g-solution under an appropriate measure. Instead of successively solving two BSDEs (one for y^0, then one for y which depends on z^0), one can first do a measure change and then solve one BSDE (for y) under the new measure. While this usually does not reduce the difficulty of the problem, it still gives a conceptually clearer view.

Example 7.19 Let the DMCUF Φ be described by the g-solution with driver $g_t(z) := -\frac{\gamma}{2}\|z\|^2$. Then with $\theta_t := \sigma_t^*(\sigma_t\sigma_t^*)^{-1}\mu_t$, we have from (7.22)

$$\begin{aligned}\hat{g}_t(z) &= \sup_{v\in\mathbb{R}^n}\{g_t(z+\sigma_t^*v)+g_t^m(-\sigma_t^*v)\}\\ &= \sup_{v\in\mathbb{R}^n}\left\{-\frac{\gamma}{2}\|z+\sigma_t^*v\|^2+v^*\mu_t\right\}\\ &= \sup_{v\in\mathbb{R}^n}\left\{-\frac{\gamma}{2}\|z+\sigma_t^*v\|^2+(\sigma_t^*v)^*\theta_t\right\}.\end{aligned}$$

By completion of the square we can rewrite the term in brackets on the RHS as

$$-\frac{\gamma}{2}\left\|\sigma_t^*v-\left(-z+\frac{1}{\gamma}\theta_t\right)\right\|^2-z^*\theta_t+\frac{1}{2\gamma}\|\theta_t\|^2$$

to get

$$\hat{g}_t(z) = -\frac{\gamma}{2}\left\|\mathbf{\Pi}_t\left(-z+\frac{1}{\gamma}\theta_t\right)\right\|^2-z^*\theta_t+\frac{1}{2\gamma}\|\theta_t\|^2,$$

where $\mathbf{\Pi}_t(u)$ denotes the projection of u onto the orthogonal complement of $\sigma_t^*(\mathbb{R}^n)$. Denoting by $(\,.\,,\,.\,)$ the scalar product and using the properties $\|a\|^2-\|b\|^2=\|a-b\|^2+2(a-b\,,b)$ and $\big(\mathbf{\Pi}_t(a)\,,\mathbf{\Pi}_t(b)\big)=\big(a\,,\mathbf{\Pi}_t(b)\big)$ and linearity of the projection $\mathbf{\Pi}_t$, we get

$$\begin{aligned}\tilde{\hat{g}}_t(z) &= \hat{g}_t(z+z_t^0)-\hat{g}_t(z_t^0)\\ &= -\frac{\gamma}{2}\left(\left\|\mathbf{\Pi}_t\left(-z-z_t^0+\frac{1}{\gamma}\theta_t\right)\right\|^2-\left\|\mathbf{\Pi}_t\left(-z_t^0+\frac{1}{\gamma}\theta_t\right)\right\|^2\right)-z^*\theta_t\\ &= -\frac{\gamma}{2}\|\mathbf{\Pi}_t(z)\|^2-\gamma\left(-\mathbf{\Pi}_t(z)\,,\frac{1}{\gamma}\mathbf{\Pi}_t\left(-\gamma z_t^0+\theta_t\right)\right)-z^*\theta_t\\ &= -\frac{\gamma}{2}\|\mathbf{\Pi}_t(z)\|^2+\left(z\,,\mathbf{\Pi}_t\left(-\gamma z_t^0+\theta_t\right)\right)-z^*\theta_t\\ &= -\frac{\gamma}{2}\|\mathbf{\Pi}_t(z)\|^2+\left(z\,,\mathbf{\Pi}_t\left(-\gamma z_t^0+\theta_t\right)-\theta_t\right).\end{aligned}$$

In particular, if the process $\mathcal{E}(\int\theta_s^0\,dW_s)$ for $\theta_t^0 := \mathbf{\Pi}_t\left(-\gamma z_t^0+\theta_t\right)-\theta_t$ is the density process of some equivalent martingale measure $\mathbb{Q}^0\in\mathcal{M}_1^e(\mathbb{P})$, then we obtain

$$\begin{aligned}-dp_t(X) &= -\frac{\gamma}{2}\|\mathbf{\Pi}_t(z_t)\|^2+\left\langle z_t,\theta_t^0\right\rangle dt-z_t^*\,dW_t\\ &= -\frac{\gamma}{2}\|\mathbf{\Pi}_t(z_t)\|^2-z_t^*\,dW_t^0,\\ p_T(X) &= X,\end{aligned}$$

where $W^0 := W-\int\theta_s^0\,ds$ is a Brownian motion under $\mathbb{Q}^0$. This representation has the advantage that the driver does not depend on z^0; it was presented (in a more general setting) by Rouge and El Karoui in Theorem 5.1 in [REK00]. To see that their results agree with ours, note that the price/value process in [REK00] is the seller price process whereas we consider the value process for the buyer. Moreover, our process z^0 is associated to the g-solution which describes the process $U^{\text{opt}}(0) = (-\frac{1}{\gamma}\operatorname{ess\,inf}_{h\in\mathcal{C}_t}\mathbb{E}[\exp(-\gamma h)|\mathcal{F}_t])_{0\leq t\leq T}$

whereas their process z^0 is associated to the g-solution which describes $-U^{\text{opt}}(0)$. However, one can easily check that if (y, z) denotes the solution for a driver g and terminal condition $-X$ and if $(\tilde{y}, \tilde{z})$ denotes the solution for the driver $\tilde{g}_t(y, z) := -g_t(-y, -z)$ and terminal condition X, then $(\tilde{y}, \tilde{z}) = (-y, -z)$. Therefore the driver in [REK00] should be compared with $-\tilde{\hat{g}}_t(-\,.\,)$ where in addition z^0 is replaced by $-z^0$. The BSDE for U^{opt} can also be found in Theorem 7 of [HIM05]. For similar reasons as above, the driver there should be compared with $\hat{g}(-\,.\,)$.

Remark 7.20 Although the driver $\hat{g}$ in the above example is not deterministic, the corresponding utility indifference price satisfies $(\mathcal{R})$; see Proposition 15 in [MS05]. It would be interesting to see an explanation for why this happens. $\Diamond$

7.2 Example 2.

In this example we show that an MCohUF at time 0 cannot always be extended to a time-consistent normalized DMCUF; note that if there exists any time-consistent extension, then there also exists a normalized extension. More precisely, we consider the MCohUF

$$\Phi_0(X) := \mathbb{E}[X] - a\|(X - \mathbb{E}[X])^-\|_p \quad \text{for } X \in \mathbf{L}^\infty, \tag{7.29}$$

where $0 < a \leq 1$ is a constant and $\|\,.\,\|_p$ is the $\mathbf{L}^p$-norm for some $1 \leq p < \infty$. One straightforward extension to a DMCohUF can be obtained by setting

$$\Phi_t(X) := \mathbb{E}[X|\mathcal{F}_t] - a\left(\mathbb{E}\left[(\,(X - \mathbb{E}[X|\mathcal{F}_t])^-)^p\,|\mathcal{F}_t\right]\right)^{\frac{1}{p}}, \quad 0 \leq t \leq T.$$

Then for each time t we can specify a convex $\mathbf{L}^1$-closed set $\mathcal{Q}_t$ of measures representing Φ_t as in (3.10) of Theorem 3.16. However, we show by a counterexample that $\mathcal{Q}_0$ is in general not weakly m-stable so that by Lemma 3.29 Φ is not time-consistent. Moreover, we also show that it is even impossible to extend Φ_0 to any time-consistent DMCUF at all. The point of this example is to illustrate that time-consistency is a rather severe condition on a DMCUF.

The definition of Φ_0 is inspired by an example given in [Fis01] by Fischer who considers (static) coherent risk measures depending on one-sided moments. It is quite natural to define an MCohUF in this way, since it is just the expected value of the payoff minus a term which punishes the downside risk.

Let us first show that at each time $t \in [0, T]$, $\Phi_t(\,.\,)$ can be represented as

$$\Phi_t(X) \;=\; \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{Q}_t^e} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t], \tag{7.30}$$

where for $p > 1$

$$\mathcal{Q}_t := \left\{ \mathbb{Q} \in \mathcal{M}_1(\mathbb{P}) \,\middle|\, \frac{d\mathbb{Q}}{d\mathbb{P}} = 1 + a(Y - \mathbb{E}[Y|\mathcal{F}_t]),\, Y \geq 0,\, \mathbb{E}[Y^q|\mathcal{F}_t] \leq 1 \right\}$$

with q conjugate to p, and for $p = 1$

$$\mathcal{Q}_t := \left\{ \mathbb{Q} \in \mathcal{M}_1(\mathbb{P}) \,\middle|\, \frac{d\mathbb{Q}}{d\mathbb{P}} = 1 + a(Y - \mathbb{E}[Y|\mathcal{F}_t]),\, 0 \leq Y \leq 1 \right\}.$$

Note that by Example 3.3 b), this shows in particular that Φ is a DMCohUF. For $t = 0$ the proof of (7.30) can be found in [Del00], and for general $t \in [0, T]$, it works similarly as follows. Fix $X \in \mathbf{L}^\infty$ and $t \in [0, T]$. We start with the case when $p > 1$ and define

$$\tilde{Y} := \frac{\left((X - \mathbb{E}[X|\mathcal{F}_t])^+\right)^{p-1}}{\left(\mathbb{E}\left[\left((X - \mathbb{E}[X|\mathcal{F}_t])^+\right)^p \Big| \mathcal{F}_t\right]\right)^{\frac{p-1}{p}}} \geq 0.$$

Then $\mathbb{E}\big[\tilde{Y}^q \big| \mathcal{F}_t\big] = 1$ and hence also $\mathbb{E}\big[\tilde{Y}\big|\mathcal{F}_t\big] \leq 1$ by the conditional Jensen inequality. Denote by $\tilde{Z}$ the density process of the corresponding measure $\tilde{\mathbb{Q}}$ in $\mathcal{Q}_t$ so that $\tilde{Z}_s := \mathbb{E}\left[1 + a\left(\tilde{Y} - \mathbb{E}[\tilde{Y}|\mathcal{F}_t]\right) \Big| \mathcal{F}_s\right]$ for $s \in [0, T]$. Note that $\tilde{Z}_t = 1$. Since $\tilde{Y} - \mathbb{E}[\tilde{Y}|\mathcal{F}_t] \geq -1$, $\tilde{\mathbb{Q}}$ is equivalent to $\mathbb{P}$ for $a < 1$. If $a = 1$, then $\tilde{\mathbb{Q}}$ need not be absolutely continuous with respect to $\mathbb{P}$. However, we shall see that $\mathcal{Q}_t$ is convex and contains $\mathbb{P}$, so that we can approximate $\tilde{\mathbb{Q}}$ in $\mathbf{L}^1(\mathbb{P})$ by the sequence $(\mathbb{Q}^n)_{n \in \mathbb{N}} \subseteq \mathcal{Q}^e_t$ associated to the sequence of densities $Z^\epsilon_T := \epsilon + (1 - \epsilon)\tilde{Z}_T$, $0 < \epsilon < 1$; see the proof of Lemma 3.29. Since $\tilde{Z}_t = 1$, we have

$$\begin{aligned}
\mathbb{E}\left[\frac{\tilde{Z}_T}{\tilde{Z}_t} X \middle| \mathcal{F}_t\right] &= \mathbb{E}[X|\mathcal{F}_t] + \mathbb{E}\left[\tilde{Z}_T\left(X - \mathbb{E}[X|\mathcal{F}_t]\right) \Big| \mathcal{F}_t\right] \\
&= \mathbb{E}[X|\mathcal{F}_t] + \mathbb{E}\left[(\tilde{Z}_T - 1 + a\mathbb{E}[\tilde{Y}|\mathcal{F}_t])\left(X - \mathbb{E}[X|\mathcal{F}_t]\right) \Big| \mathcal{F}_t\right] \\
&= \mathbb{E}[X|\mathcal{F}_t] + \mathbb{E}\left[a\tilde{Y}\left(X - \mathbb{E}[X|\mathcal{F}_t]\right) \Big| \mathcal{F}_t\right] \qquad (7.31) \\
&= \mathbb{E}[X|\mathcal{F}_t] + a\frac{\mathbb{E}\left[\left((X - \mathbb{E}[X|\mathcal{F}_t])^+\right)^{p-1}\left(X - \mathbb{E}[X|\mathcal{F}_t]\right) \Big| \mathcal{F}_t\right]}{\left(\mathbb{E}\left[\left((X - \mathbb{E}[X|\mathcal{F}_t])^+\right)^p \Big| \mathcal{F}_t\right]\right)^{\frac{p-1}{p}}} \\
&= \mathbb{E}[X|\mathcal{F}_t] + a\left(\mathbb{E}\left[\left((X - \mathbb{E}[X|\mathcal{F}_t])^+\right)^p \Big| \mathcal{F}_t\right]\right)^{\frac{1}{p}}. \qquad (7.32)
\end{aligned}$$

Now take $\mathbb{Q}' \in \mathcal{Q}_t$ with corresponding Y' and denote the density process of $\mathbb{Q}'$ by Z'. As above we obtain

$$\begin{aligned}
\mathbb{E}\left[\frac{Z'_T}{Z'_t} X \middle| \mathcal{F}_t\right] &= \mathbb{E}[X|\mathcal{F}_t] + \mathbb{E}\left[aY'\left(X - \mathbb{E}[X|\mathcal{F}_t]\right) \big| \mathcal{F}_t\right] \\
&\leq \mathbb{E}[X|\mathcal{F}_t] + a\mathbb{E}\left[Y'\left(X - \mathbb{E}[X|\mathcal{F}_t]\right)^+ \Big| \mathcal{F}_t\right] \\
&\leq \mathbb{E}[X|\mathcal{F}_t] + a\left(\mathbb{E}\left[(Y')^q \big| \mathcal{F}_t\right]\right)^{\frac{1}{q}} \left(\mathbb{E}\left[\left((X - \mathbb{E}[X|\mathcal{F}_t])^+\right)^p \Big| \mathcal{F}_t\right]\right)^{\frac{1}{p}} \\
&\leq \mathbb{E}[X|\mathcal{F}_t] + a\left(\mathbb{E}\left[\left((X - \mathbb{E}[X|\mathcal{F}_t])^+\right)^p \Big| \mathcal{F}_t\right]\right)^{\frac{1}{p}}
\end{aligned}$$

by using Hölder's inequality and the definition of $\mathcal{Q}_t$. Replacing X by $\hat{X} := -X$ and using $(\hat{X} - \mathbb{E}[\hat{X}|\mathcal{F}_t])^+ = (X - \mathbb{E}[X|\mathcal{F}_t])^-$ gives after changing signs that

$$\mathbb{E}\left[\frac{Z'_T}{Z'_t}X\middle|\mathcal{F}_t\right] \geq \mathbb{E}[X|\mathcal{F}_t] - a\left(\mathbb{E}\left[\left((X - \mathbb{E}[X|\mathcal{F}_t])^-\right)^p \,\middle|\, \mathcal{F}_t\right]\right)^{\frac{1}{p}} = \Phi_t(X).$$

Analogously, (7.32) can be transformed into

$$\mathbb{E}\left[\frac{\tilde{Z}_T}{\tilde{Z}_t}X\middle|\mathcal{F}_t\right] = \Phi_t(X).$$

This proves (7.30) for $p > 1$. If $p = 1$ we take $\tilde{Y} := \mathbf{1}_{\{X<\mathbb{E}[X|\mathcal{F}_t]\}}$ and obtain as in (7.31) that

$$\begin{aligned}
\mathbb{E}\left[\frac{\tilde{Z}_T}{\tilde{Z}_t}X\middle|\mathcal{F}_t\right] &= \mathbb{E}[X|\mathcal{F}_t] + a\mathbb{E}\left[\tilde{Y}\left(X - \mathbb{E}[X|\mathcal{F}_t]\right)\,\middle|\,\mathcal{F}_t\right] \\
&= \mathbb{E}[X|\mathcal{F}_t] - a\mathbb{E}\left[(X - \mathbb{E}[X|\mathcal{F}_t])^-\,\middle|\,\mathcal{F}_t\right] \\
&= \Phi_t(X)
\end{aligned}$$

and that for arbitrary $\mathbb{Q}' \in \mathcal{Q}_t$ with corresponding Y' and density Z', we have

$$\begin{aligned}
\mathbb{E}\left[\frac{Z'_T}{Z'_t}X\middle|\mathcal{F}_t\right] &\leq \mathbb{E}[X|\mathcal{F}_t] + a\mathbb{E}\left[Y'\left(X - \mathbb{E}[X|\mathcal{F}_t]\right)^+\,\middle|\,\mathcal{F}_t\right] \\
&\leq \mathbb{E}[X|\mathcal{F}_t] + a\mathbb{E}\left[(X - \mathbb{E}[X|\mathcal{F}_t])^+\,\middle|\,\mathcal{F}_t\right].
\end{aligned}$$

The same arguments as above then again yield (7.30).

In a second step, we now prove that $\mathcal{Q}_t$ is convex and closed in $\mathbf{L}^1$. Convexity is easy since for $p > 1$, the boundedness by 1 of $\mathbb{E}\left[(\alpha Y + (1-\alpha)Y')^q\middle|\mathcal{F}_t\right]$ follows from the conditional Minkowski inequality. To show closedness, we fix a sequence $(\mathbb{Q}_n)_{n\in\mathbb{N}} \subseteq \mathcal{Q}_t$ which converges in $\mathbf{L}^1$ to some $\overline{\mathbb{Q}}$ and denote by Z^n_T and $\overline{Z}_T$ their respective densities and by $(Y_n)_{n\in\mathbb{N}}$ and $\overline{Y}$ the associated random variables from the definition of $\mathcal{Q}_t$. Since each $f_n := \mathbb{E}[Y_n|\mathcal{F}_t]$ satisfies $0 \leq f_n \leq 1$, Lemma 3.2 of [Sch92] ensures the existence of a sequence $(\hat{f}_n)_{n\in\mathbb{N}}$ of convex combinations $\hat{f}_n \in \operatorname{conv}\{f_n, f_{n+1}, \ldots\}$ which converges to some $\hat{f}$ almost surely and hence also in $\mathbf{L}^1$. Denote for each $n \in \mathbb{N}$ by $\hat{Z}^n_T \in \operatorname{conv}\{Z^n_T, Z^{n+1}_T, \ldots\}$ and $\hat{Y}_n \in \operatorname{conv}\{Y_n, Y_{n+1}, \ldots\}$ the convex combinations with the same weights as $\hat{f}_n$. Then

$$\begin{aligned}
\mathbb{E}[\,|\hat{Y}_m - \hat{Y}_n|\,] &\leq \mathbb{E}\left[\left|(\hat{Y}_m - \hat{f}_m) - (\hat{Y}_n - \hat{f}_n)\right|\right] + \mathbb{E}\left[\left|\hat{f}_m - \hat{f}_n\right|\right] \\
&= \mathbb{E}\left[\left|\frac{1}{a}(\hat{Z}^m_T - 1) - \frac{1}{a}(\hat{Z}^n_T - 1)\right|\right] + \mathbb{E}\left[\left|\hat{f}_m - \hat{f}_n\right|\right],
\end{aligned}$$

and the RHS converges to 0 for $m, n \to \infty$ since $(\hat{Z}^n_T)_{n\in\mathbb{N}}$ converges like $(Z^n_T)_{n\in\mathbb{N}}$ to $\overline{Z}_T$ in $\mathbf{L}^1$; this uses that $\hat{Z}^n_T \in \operatorname{conv}\{Z^n_T, Z^{n+1}_t, \ldots\}$. Thus the Cauchy sequence $(\hat{Y}_n)_{n\in\mathbb{N}}$ converges to some $\hat{Y} \geq 0$ in $\mathbf{L}^1$. If $p > 1$, the conditional Minkowski inequality implies that $\mathbb{E}[\hat{Y}_n^q|\mathcal{F}_t] \leq 1$ for each $n \in I\!N$ and hence by Fatou's lemma also $\mathbb{E}[\hat{Y}^q|\mathcal{F}_t] \leq 1$; for $p = 1$, we have $0 \leq \hat{Y} \leq 1$. Moreover,

$$\hat{Z}_T^n = 1 + a\left(\hat{Y}_n - \mathbb{E}\left[\hat{Y}_n\middle|\mathcal{F}_t\right]\right) \geq 0$$

converges for $n \to \infty$ in $\mathbf{L}^1$ to

$$\hat{Z}_T := 1 + a\left(\hat{Y} - \mathbb{E}\left[\hat{Y}\middle|\mathcal{F}_t\right]\right)$$

since $\hat{Y}_n \to \hat{Y}$ in $\mathbf{L}^1$. So $\hat{Z}_T$ is ≥ 0 and the density of an element of $\mathcal{Q}_t$. But we already know that $\hat{Z}_T^n \to \overline{Z}_T$ in $\mathbf{L}^1$; hence $\overline{Z}_T = \hat{Z}_T$ which implies that $\mathcal{Q}_t$ is closed.

Finally we provide a counterexample which shows that Φ is in general not a time-consistent DMCohUF and that it is even impossible to redefine it for $t \in (0, T]$ such that Φ becomes a time-consistent DMCUF. In fact, the counterexample shows that $\mathcal{Q}_0$ is not weakly m-stable in general, which is by Lemma 3.29 a necessary condition for time-consistency if Φ_0 is positively homogeneous; note that we showed in the proof of Lemma 3.29 that the $\mathbf{L}^1$-closed convex st $\mathcal{Q}_0$ representing a DMCohUF at time 0 is unique.

Counterexample: Let $\Omega = \{\omega_1, \ldots, \omega_6\}$, $\mathcal{F}$ the power set of Ω, $T = 2$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \sigma(\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\})$, $\mathcal{F}_2 = \mathcal{F}$, $a = p = 1$, $p_i := \mathbb{P}[\{w_i\}]$, $i = 1, \ldots, 6$, $p_1 = 100p_2$, $p_6 = 100p_5$. Define two densities Z_T^1 and Z_T^2 of elements of $\mathcal{Q}_0$ by their associated random variables Y^1 and Y^2, where $\mathbb{E}[Y^i] = \frac{1}{2}$, $0 \leq Y^i \leq 1$, $i = 1, 2$, and

$$Y^1(\omega_1) = \frac{1}{100}, \quad Y^1(\omega_2) = 1, \quad Y^1(\omega_5) = 1, \quad Y^1(\omega_6) = 1,$$

$$Y^2(\omega_1) = \frac{1}{100}, \quad Y^2(\omega_2) = 1, \quad Y^2(\omega_5) = 1, \quad Y^2(\omega_6) = \frac{1}{100}.$$

If $\mathcal{Q}_0$ is weakly m-stable, then

$$\begin{aligned}
\overline{Z}_T &:= Z_1^1 \frac{Z_T^2}{Z_1^2} \\
&= \left(1 + (\mathbb{E}[Y^1|\mathcal{F}_1] - \mathbb{E}[Y^1])\right) \frac{1 + (Y^2 - \mathbb{E}[Y^2])}{1 + (\mathbb{E}[Y^2|\mathcal{F}_1] - \mathbb{E}[Y^2])} \\
&= 1 + \left(-1 + \left(1 + (\mathbb{E}[Y^1|\mathcal{F}_1] - \mathbb{E}[Y^1])\right) \frac{1 + (Y^2 - \mathbb{E}[Y^2])}{1 + (\mathbb{E}[Y^2|\mathcal{F}_1] - \mathbb{E}[Y^2])}\right)
\end{aligned}$$

must be the density of some element of $\mathcal{Q}_0$. Since

$$\tilde{Y} := \left(1 + (\mathbb{E}[Y^1|\mathcal{F}_1] - \mathbb{E}[Y^1])\right) \frac{1 + (Y^2 - \mathbb{E}[Y^2])}{1 + (\mathbb{E}[Y^2|\mathcal{F}_1] - \mathbb{E}[Y^2])}$$

has $\mathbb{E}[\tilde{Y}] = 1$ we can write

$$\overline{Z}_T = 1 + \left((\tilde{Y} + c) - \mathbb{E}[\tilde{Y} + c]\right)$$

for any $c \in I\!R$, and this is the unique decomposition of the form "$1 + (Y - \mathbb{E}[Y])$", where Y is a random variable, except for the constant c. $\overline{Z}_T$ is an element of $\mathcal{Q}_0$ if and only if there exists $c \in I\!R$ such that $0 \leq \tilde{Y} + c \leq 1$. Since $\tilde{Y} \geq 0$ this is equivalent to

$$\max_{i \in \{1,\dots,6\}} \tilde{Y}(\omega_i) - \min_{i \in \{1,\dots,6\}} \tilde{Y}(\omega_i) \leq 1. \tag{7.33}$$

However, $\mathbb{E}[Y^1|\mathcal{F}_1](\omega_1) = \mathbb{E}[Y^2|\mathcal{F}_1](\omega_1)$ so that

$$\tilde{Y}(\omega_1) = 1 + Y^2(\omega_1) - \mathbb{E}\left[Y^2\right] = 0.51$$

and $\mathbb{E}[Y^1|\mathcal{F}_1](\omega_5) = 1$ and $\mathbb{E}[Y^2|\mathcal{F}_1](\omega_5) = \frac{2}{101}$ imply that

$$\tilde{Y}(\omega_5) = \left(1 + 1 - \frac{1}{2}\right) \frac{1 + 1 - \frac{1}{2}}{1 + \frac{2}{101} - \frac{1}{2}} = \frac{303}{70}.$$

Therefore (7.33) is not satisfied and $\mathcal{Q}_0$ is not weakly m-stable. ♢

8 Conclusion

We investigate the dynamic behaviour of a continuous-time valuation functional obtained by utility indifference. The underlying preference structure is given by a monetary concave utility functional (MCUF) which is minus a conditional convex risk measure. The main tool for the analysis of the valuation functional is the convolution of abstract dynamic MCUFs. This allows us to obtain properties for the MCUF, which assigns to each contingent claim the maximal attainable utility of the agent when she can trade in a financial market. After a normalization, this produces the utility indifference valuation functional p. We give a dual representation for p and show that it is strongly time-consistent if the original preference structure is time-consistent and if the investment opportunities change nicely over time. Moreover, we give conditions when p is consistent with the no-arbitrage principle so that it could also be regarded as a price for a contingent claim, and we discuss its relation to good deal bounds. Finally we recall the connection between solutions of backward stochastic differential equations (BSDEs) and dynamic MCUFs and illustrate in an example how p can be described by a BSDE when the MCUF which corresponds to the agent's preference structure is specified by a BSDE.

This paper deals only with valuations for payoffs due at one fixed time in the future. It would be interesting to generalize the results to payoff streams. In view of recent research about dynamic risk measures for processes, this appears to be a very promising project.

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